Warped-Time Methods

Yielding New Periodic Solutions of an Expanded van der Pol (Liénard) Equation

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Chapter 1. Van der Pol Equations

Equation Chapter 1 Section 1

1.1 THE EXTENDED VAN DER POL EQUATION

Balthasar van der Pol's equation

$$\ddot{u}(t)+\varepsilon\left[u^2(t)-1\right]\dot{u}(t)+u(t)=0\ , \tag{1.1-1}$$

subsequently cited as the LvdP [Legacy van der Pol] equation, is the simplest, most familiar, and most thoroughly analyzed member in the family of Extended van der Pol [XvdP] equations (sometimes cited as the Generalized Liénard Equation) having the form

$$\ddot{u}(t)+\varepsilon\cdot F[u(t);\dot{u}(t)]\cdot\dot{u}(t)+G[u(t)]=0\ . \tag{1.1-2}$$

Our particular interest is the solution[1] of eq. (1.1-2) exhibiting periodic motion.

Although van der Pol proposed the LvdP as a simple nonlinear equation whose solutions displayed spontaneous oscillations resembling those observed previously within various electrical mechanisms [GINOUX AND LETELLIER [2012], we prefer to interpret the XvdP in mechanical terms as the *equation-of-motion* governing the *displacement* u of a *unit point mass* with momentum $p\equiv\dot{u}$ as a function of an *elapsed-time* t, subject to a viscous *Dissipation Force* $\varepsilon F[u;p]p$ and a *Restoring Force* $G[u]$; we further presume (without loss of generality) hat the Dissipation and Restoring Forces are configured to support the constraints $\varepsilon>0$ and $\dot{u}(0)>0$.

Our subsequent XvdP analyses use lower case letters to denote a general solution while reserving upper case letters to denote a periodic solution; they also append the XvdP's nonlinearity parameter value as a subscript, while using the *exceptional* notation

$$\lim_{\varepsilon\to 0^+} u_\varepsilon(t) \square\ u_0(t) \tag{1.1-3}$$

and addressing the actual absence of a Dissipation Force by replacing the XvdP eq. (1.1-2) with the *Null Form* (NF) equation

[1] We use of singular terminology simply as a device to simply grammar without precluding the existence of multiple XvdP solutions with difference parameter values.

$$\ddot{u}_N(t) + G[u_N(t)] = 0 \quad . \tag{1.1-4}$$

XVDP ENERGY REPRESENTATION

Upon multiplying each term of eq. (1.1-4) by the NF momentum $\dot{u}_N(t)$ and integrating with respect to elapsed-time, we use the classical mechanics definitions

$$\text{Kinetic Energy}: \ \mathrm{T}[p] \equiv p^2/2 \ ,$$
$$\text{Potential Energy}: \ \mathrm{V}[u] \equiv \int_0^u d\xi \ G[\xi] \ ,$$

to recover the XvdP's NF energy representation (also cited as the *First Integral*) in the form

$$\mathrm{T}[p_N(t)] + \mathrm{V}[u_N(t)] = \mathrm{E}_{TOT} \, , \tag{1.1-5}$$

with E_{TOT} as the (constant) Total NF energy. In the same way, upon multiplying each term of eq. (1.1-2) by the XvdP momentum $\dot{u}(t)$ and integrating with respect to elapsed-time, we use the new definition

$$\text{Dissipation Energy}: \ \mathrm{E}_D[t; u, p] \equiv \int_0^t ds \ F[u(s); p(s)] \cdot p^2(s) \ ,$$

to recover the XvdP's energy representation in the form

$$\mathrm{T}[p_\varepsilon(t)] + \mathrm{V}[u_\varepsilon(t)] = \mathrm{E}_I - \varepsilon \cdot \mathrm{E}_D[t; u_\varepsilon, p_\varepsilon] \ . \tag{1.1-6}$$

with $\mathrm{E}_I = \mathrm{T}[p_\varepsilon(0)] + \mathrm{V}[u_\varepsilon(0)]$ as the (constant) Initial XvdP energy.

PERIODIC XVDP SOLUTIONS

Classical Mechanics [GOLDSTEIN, 1951] describes two kinds of periodic motion., with the more common kind, subsequently cited as *Libration,* occurring when both the displacement and momentum are periodic functions of time sharing a common period T_P, i.e.

$$\begin{aligned} u_\varepsilon(t + T_P) &= u_\varepsilon(t) \ , \\ p_\varepsilon(t + T_P) &= p_\varepsilon(t) \ ; \end{aligned} \tag{1.1-7}$$

with this kind of periodic motion, the displacement and momentum continually retrace a fixed Limit-Cycle orbit in (u, p) phase-space as the elapsed-time t increases. A less common kind of periodic motion, most often associated with that of rigid bodies and subsequently cited as *Rotation*, occurs when the displacement is a continuously increasing an angle-of-rotation (observable only mod 2π) about a fixed axis, with eq.(1.1-7) replaced by

$$\begin{aligned} u_\varepsilon(t+T_P) &= u_\varepsilon(t) + 2\pi \ , \\ p_\varepsilon(t+T_P) &= p_\varepsilon(t) \ , \end{aligned} \tag{1.1-8}$$

and the *observable* angle of rotation satisfying

$$\begin{aligned} \tilde{u}_\varepsilon(t+T_P) &\equiv u_\varepsilon(t+T_P)\big|_{2\pi} = u_\varepsilon(t)\big|_{2\pi} = \tilde{u}_\varepsilon(t) \ , \\ p_\varepsilon(t+T_P) &= \dot{u}_\varepsilon(t+T_P) = \dot{u}_\varepsilon(t) = p_\varepsilon(t) \ ; \end{aligned} \tag{1.1-9}$$

with this kind of periodic motion, the momentum becomes a periodic function

$$p_\varepsilon[u_\varepsilon(t+T_P)] = p_\varepsilon[u_\varepsilon(t) + 2\pi] \tag{1.1-10}$$

of the displacement (u, p) phase-space as the elapsed-time t increases without a Limit-Cycle orbit. Our subsequent analyses accommodate periodic motion exhibiting *either* Libration or Rotation by using their common definition of periodic momentum; in particular, we impose the initial conditions

$$\begin{aligned} u_\varepsilon(0) &= 0 \ , \\ p_\varepsilon(0) &= p_I > 0 \ , \end{aligned} \tag{1.1-11}$$

and ordinarily characterize a periodic solution of eq. (1.1-2) by[2] its Initial Momentum $P_0[\varepsilon] \equiv p_I$ and Period $T_0[\varepsilon] = T_P$.

It further follows from eqs. (1.1-7) and (1.1-8) that Kinetic Energy, *which depends only upon momentum*, must satisfy the condition

$$\mathrm{T}[p_\varepsilon(t+T_P)] - \mathrm{T}[p_\varepsilon(t)] = 0 \ . \tag{1.1-12}$$

[2] when periodic motion exhibits Libration, we may provide the corresponding Maximum Displacement $U_0[\varepsilon]$ to facilitate comparison with other analyses.

In the same way, the Potential Energy, *which depends only upon the observable displacement*, must satisfy the condition

$$\mathrm{V}[u_\varepsilon(t+T_P)] - \mathrm{V}[u_\varepsilon(t)] = 0 \quad . \tag{1.1-13}$$

Thus, together with the definition of eq. (1.1-6), eqs. (1.1-12) and (1.1-13) impose the Dissipation Energy constraint

$$\begin{aligned} &\mathrm{E}_D[t+T_P; u_\varepsilon, p_\varepsilon] - \mathrm{E}_D[t; u_\varepsilon, p_\varepsilon] = \\ &= \int_t^{t+T_P} dt \; F[u_\varepsilon(t); p_\varepsilon(t)] \cdot p_\varepsilon^2(t) = 0 \quad . \end{aligned} \tag{1.1-14}$$

as a necessary (but not sufficient because Kinetic Energy does not include the algebraic sign of momentum) condition for the existence of an XvdP solution of eq. (1.1-2) exhibiting periodic motion.

Equation Section (Next)

1.2 THE LEGACY VAN DER POL EQUATION

Using a variety of analytic methods, van der Pol and others [see Annex A] then characterized the LvdP in terms of its Maximum Displacement $U_0[\varepsilon]$ and Period $T_0[\varepsilon]$ for various values of the nonlinearity parameter ε; subsequent numerical evaluations [KROGDAHL, 1960; ZONNEVELD, 1966] yielded the results shown in Table 1.1-1.

Table 1.1-1 – LvdP Period $T_0[\varepsilon]$ and Maximum Displacement $U_0[\varepsilon]$ for Representative Values of the Nonlinearity Parameter ε

ε	$T_0[\varepsilon]$	$U_0[\varepsilon]$	ε	$T_0[\varepsilon]$	$U_0[\varepsilon]$
0	6.28319	2.00000	10	19.07837	2.01429
1	6.66329	2.00862	20	34.68232	2.00779
2	7.62987	2.01989	30	50.54369	2.00516
3	8.85909	2.02330	40	66.50137	2.00379
4	10.20352	2.02296	50	82.50833	2.00296
5	11.71223	2.02151	60	98.54479	2.00240
6	13.06187	2.01983	70	114.60067	2.00201
7	14.53975	2.01822	80	130.67020	2.00172
8	16.03818	2.01675	90	146.74979	2.00150
9	17.55218	2.01544	100	162.83707	2.00132

VAN DER POL [1926] also used the explicit absence of elapsed-time t in eq. **Error! Reference source not found.** to express momentum as a function $p[u]$ of displacement satisfying

$$p[u] \cdot \frac{dp[u]}{du} = \varepsilon \cdot (1 - u^2) \cdot p[u] - u \ , \tag{1.2-1}$$

suitable for graphical solution via the Method of Isoclines [HAYASHI, 1964]; he then obtains LvdP displacements $u_\varepsilon(t)$ for the particular nonlinearity parameter values $\varepsilon = 0.1$, $\varepsilon = 1.0$, and $\varepsilon = 10.0$, while further observing that, for each of these nonlinearity parameter values, the asymptotic (u, p) phase-space trajectories approach a closed *Limit-Cycle* orbit, thereby permitting recovery of the periodic LvdP solutions shown in Fig. 1.1-1.

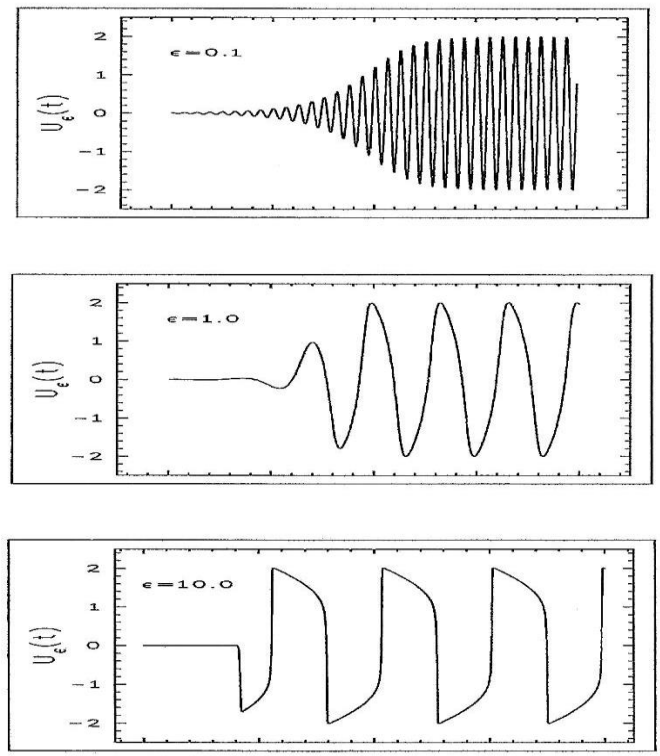


Figure 1.1 – Period solutions of the LvdP for representative nonlinear parameter values $\varepsilon = 0.1$, 1.0, and 10.0

Equation Section (Next)

1.3 THE GENERALIZED VAN DER POL EQUATION

The Generalized van der Pol [GvdP]equation

(1.3-1) $$\ddot{u} + \varepsilon \cdot [u^{2d} - 1] \cdot \dot{u}^{2m+1} + u = 0$$

has a NF Energy equation

(1.3-2) $$p_N^2(t, p_I) + u_N^2(t, p_I)] = p_I^2$$

whose family of periodic NF solutions

(1.3-3) $$\begin{aligned} u_N(t; p_I) &= p_I \cdot Sin(t)\,, \\ p_N(t; p_I) &= p_I \cdot Cos(t)\,, \end{aligned}$$

have a common, full-cycle period $T_P = 2\pi$. Thus, a particular NF solution $u_N(t, p_*)$ will also be the periodic WD GvdP solution $U_0(t)$ satisfying eq. (1.3-1) when it also satisfies the Dissipation Energy constraint

(1.3-4) $$\int_0^{2\pi} dt \left[[p_*^2 \cdot Sin^2(t)]^d - 1 \right] \cdot [Cos^2(t)]^{m+1} = 0\,,$$

imposed by eq. (1.1-14), i.e. when

(1.3-5) $$p_*[d, m] = \left[\frac{\int_0^{2\pi} dt\ Cos^{2m+2}(t)}{\int_0^{2\pi} dt\ Sin^{2d}(t) \cdot Cos^{2m+2}(t)} \right]^{\frac{1}{2d}}.$$

The Beta function definition

(1.3-6) $$\mathrm{B}(a, b) \equiv 2 \cdot \int_0^{\pi/2} ds\ Sin^{2a-1}(s) \cdot Cos^{2b-1}(s) = \frac{\Gamma(a) \cdot \Gamma(b)}{\Gamma(a+b)}\,,$$

then allows us to express eq. (1.3-5) as

$$P_0[0;d,m] \equiv p_*[d,m] = \left[\frac{\Gamma(½)}{\Gamma(d+½)} \cdot \frac{\Gamma(d+m+2)}{\Gamma(m+2)}\right]^{\frac{1}{2d}} =$$
$$= 2 \cdot \left[\frac{(d+m+1)!}{(m+1)!} \cdot \frac{d!}{(2d)!}\right]^{\frac{1}{2d}} , \tag{1.3-7}$$

with the special result

$$P_0[0;m+1,m] = 2 \quad . \tag{1.3-8}$$

Table 2.2-1 provides the WD GvdP solution's Initial Momentum for representative values of its Dissipation Factor exponents d and m; it also follows from eq. (1.3-3) that the representative Initial Momentum values shown in Table 2.2-1 are numerically equal to the corresponding GvdP Maximum Amplitude values, i.e. $U_0[0;d,m] \equiv P_0[0;d,m]$.

TABLE 2.2-1 – REPRESENTATIVE VALUES OF THE WD GVDP SOLUTION'S INITIAL MOMENTUM $P_1[0;d,m] \equiv p_*[d,m]$

	$d=1$	$d=2$	$d=3$	$d=5$	$d=10$
$m=0$	**2.00000**	**1.68179**	**1.53945**	**1.37627**	**1.22962**
$m=1$	**2.44949**	**2.00000**	**1.78180**	**1.55995**	**1.34486**
$m=2$	**2.82843**	**2.27244**	**2.00000**	**1.72071**	**1.44717**
$m=3$	**3.16228**	**2.51487**	**2.19551**	**1.86607**	**1.54071**
$m=5$	**3.74166**	**2.93956**	**2.54042**	**2.12598**	**1.70953**
$m=10$	**4.89998**	**3.79766**	**3.24371**	**2.66022**	**2.06572**

Upon expressing Potential Energy $\mathrm{V}[u]=\nu^2(u)/2$ in terms of a linear Potential Factor $\nu(u)=u$, we can express eq. (1.3-1) in a 'curious-but-correct' form

$$\ddot{u}+\varepsilon\cdot[\nu^{2d}(u)-1]\cdot\dot{u}^{2m+1}+\nabla\cdot\mathrm{V}[u]=0 \tag{1.3-9}$$

with the corresponding NF Energy equation **Error! Reference source not found.**

$$p_N^2(t,p_I)+\nu^2[u_N(t,p_I)]=p_I^2 \tag{1.3-10}$$

allowing the Dissipation Energy constraint imposed by eq. (1.1-14) to assume the form

$$\int_0^{2\pi}dt\left[\nu^{2d}[u(t;p_I)]-1\right]\cdot\left[p_I^2-\nu^2[u(t;p_I)]\right]^{m+1}=0\ . \tag{1.3-11}$$

These results suggest the possible existence of additional XvdP equations with the form of eq. (1.3-9), whose solutions exhibit periodic motion in the presence of more elaborate Potential Factors.

Equation Section (Next)

1.4 THE CUBIC VAN DER POL EQUATION

A quadratic Potential Factor

$$\nu[u]=u^2\ , \tag{1.4-1}$$

converts the 'curious-but-correct' eq. (1.3-9) into the Cubic van der Pol [CvdP] equation

$$\ddot{u}+\varepsilon\cdot[u^{4d}(u)-1]\cdot\dot{u}^{2m+1}+2\cdot u^3=0\ ; \tag{1.4-2}$$

it similarly converts the NF Energy eq. (1.3-10) into a form

$$p_N^2(t;p_I)+u_N^4(t;p_I)=p_I^2 \tag{1.4-3}$$

having the family of periodic solutions, expressible in terms of Elliptic functions [see Annex A] having a fixed modulus $\sqrt{1/2}$ and $s \square \sqrt{2 \cdot p_I} \cdot t$, as

$$u_N(s; p_I) = \sqrt{\frac{p_I}{2}} \cdot sd(s, \sqrt{1/2}) \ ,$$

(1.4-4)

$$p_N(s; p_I) = p_I \cdot \frac{cd(s, \sqrt{1/2})}{dn(s, \sqrt{1/2})} \ ,$$

yielding the full-cycle period

(1.4-5)
$$T_N[p_I] = 4 \cdot \frac{\mathrm{K}(\sqrt{1/2})}{\sqrt{2 \cdot p_I}} \ .$$

A particular NF solution $u_N(t, p_*)$ will also be the periodic WD CvdP solution $U_0(t)$ satisfying eq. (1.4-2) when it satisfies the Dissipation Energy constraint

$$\int_0^{4 \cdot \mathrm{K}(\sqrt{1/2})} ds \left[[p_*^2 \cdot \frac{sd^4(s, \sqrt{1/2})}{4}]^d - 1 \right] \cdot \left[1 - \frac{sd^4(s, \sqrt{1/2})}{4} \right]^{m+1} = 0 \qquad (1.4\text{-}6)$$

imposed by eq. (1.1-14), i.e. when

$$p_*[d, m] = \left[\frac{\int_0^{4 \cdot \mathrm{K}(\sqrt{1/2})} ds \ [1 - \frac{1}{4} \cdot sd^4(s, \sqrt{1/2})]^{m+1}}{\int_0^{4 \cdot \mathrm{K}(\sqrt{1/2})} ds \ [\frac{1}{4} \cdot sd^4(s, \sqrt{1/2})]^d \cdot [1 - \frac{1}{4} \cdot sd^4(s, \sqrt{1/2})]^{m+1}} \right]^{\frac{1}{2d}} \ ; (1.4\text{-}7)$$

the definition

$$N_\ell \square \frac{1}{4 \cdot \mathrm{K}(\sqrt{1/2})} \cdot \int_0^{4 \cdot \mathrm{K}(\sqrt{1/2})} ds \ [\frac{1}{4} \cdot sd^4(s; \sqrt{1/2})]^\ell \qquad (1.4\text{-}8)$$

then permits eq. (1.4-7) to be expressed**Error! Reference source not found.** in the explicit form

$$P_0[0;d,m] = p_*[d,m] = \left[\frac{\sum_{\ell=0}^{m+1} \binom{m+1}{\ell} (-)^\ell N_\ell}{\sum_{\ell=0}^{m+1} \binom{m+1}{\ell} (-)^\ell N_{d+\ell}} \right]^{\frac{1}{2d}} , \qquad (1.4\text{-}9)$$

while the related definition

$$M_\ell \equiv \frac{1}{4 \cdot \mathrm{K}(\sqrt{1/2})} \cdot \int_0^{4 \cdot \mathrm{K}(\sqrt{1/2})} ds \; s d^{4\ell}(s;\sqrt{1/2})]^\ell , \qquad (1.4\text{-}10)$$

satisfying the recurrence relationship [BYRD & FRIEDMAN; 1961]

$$M_{4\ell} = 4 \cdot \frac{4 \cdot \ell - 3}{4 \cdot \ell - 1} \cdot M_{4\ell-4} , \qquad (1.4\text{-}11)$$

yields the corresponding recurrence relationship

$$(1.4\text{-}12) \quad N_\ell \equiv \frac{1}{4^\ell} \cdot M_{4\ell} = \frac{1}{4^{\ell-1}} \cdot \frac{4 \cdot \ell - 3}{4 \cdot \ell - 1} \cdot M_{4\ell-4} = \frac{4 \cdot \ell - 3}{4 \cdot \ell - 1} \cdot N_{\ell-1} ,$$

with eq. (1.4-8) supplying the initial value $N_0 = 1$.

Tables 2.3-1 and 2.3-3 show the periodic WD CvdP solution's Initial Momentum $P_1[0;d,m]$, Period $T_0[0;d,m]$, and Maximum Displacement $U_0[0;d,m]$ for representative values of d and m.

TABLE 2.3-1– REPRESENTATIVE VALUES OF
CVDP INITIAL MOMENTUM $P_0[0;d,m]$

	$d=1$	$d=2$	$d=3$	$d=5$	$d=10$
$m=0$	**2.64575**	**1.98098**	**1.71749**	**1.48097**	**1.27754**
$m=1$	**3.31662**	**2.39678**	**2.02848**	**1.69501**	**1.40516**
$m=2$	**3.87298**	**2.74770**	**2.29382**	**1.88005**	**1.51717**
$m=3$	**4.35890**	**3.05758**	**2.52991**	**2.04629**	**1.61900**
$m=5$	**5.19615**	**3.59699**	**2.94395**	**2.34084**	**1.80191**
$m=10$	**6.85565**	**4.67923**	**3.78271**	**2.94620**	**2.18586**

TABLE 2.3-2 REPRESENTATIVE VALUES OF
CVDP PERIOD $T_0[0;d,m]$

	$d=1$	$d=2$	$d=3$	$d=5$	$d=10$
$m=0$	**4.55926**	**5.26903**	**5.65878**	**6.09392**	**6.56121**
$m=1$	**4.07213**	**4.79023**	**5.20695**	**5.69616**	**6.25617**
$m=2$	**3.76831**	**4.47388**	**4.89653**	**5.40860**	**6.02080**
$m=3$	**3.55329**	**4.24113**	**4.66248**	**5.18243**	**5.82836**
$m=5$	**3.25333**	**3.91022**	**4.32220**	**4.48712**	**5.52464**
$m=10$	**2.83234**	**3.42833**	**3.81301**	**4.32055**	**5.01600**

TABLE 2.3-3– REPRESENTATIVE VALUES OF
CVDP MAXIMUM DISPLACEMENT $U_0[0;d,m]$

	$d=1$	$d=2$	$d=3$	$d=5$	$d=10$
$m=0$	**1.62658**	**1.40747**	**1.31053**	**1.21695**	**1.13028**
$m=1$	**1.82116**	**1.54815**	**1.42425**	**1.30193**	**1.18539**
$m=2$	**1.96799**	**1.65762**	**1.51454**	**1.37115**	**1.23173**
$m=3$	**2.08780**	**1.74859**	**1.59057**	**1.43049**	**1.27240**
$m=5$	**2.27951**	**1.89657**	**1.71579**	**1.52998**	**1.34235**
$m=10$	**2.61833**	**2.16315**	**1.94492**	**1.71645**	**1.47847**

Equation Section (Next)

1.5 THE VAN DER POL PENDULUM EQUATION

A periodic Potential Factor[3]

$$v_\gamma[u] = \frac{2}{\gamma} \cdot Sin(\frac{\gamma}{2} \cdot u) \ , \tag{1.5-1}$$

converts the 'curious-but-correct' eq. (1.3-9) into the vdPP [van der Pol Pendulum] equation

[3] This Potential Factor gives a Restoring Force $G[u] \square \nabla \cdot v_\gamma^2[u]/2 = \gamma^{-1} \cdot Sin[\gamma \cdot u]$ whose strength becomes weaker as the parameter $\gamma \in [0, \infty)$ increases in value; for this reason, subsequent references cite γ as the 'Softness' parameter.

$$\gamma \cdot \ddot{u}_\varepsilon + \varepsilon \left[[\frac{4}{\gamma^2} \cdot Sin^2(\frac{\gamma \cdot u_\varepsilon}{2}) - 1]^d \right] \cdot [\dot{u}_\varepsilon^2]^m \cdot \gamma \cdot \dot{u}_\varepsilon + Sin(\gamma \cdot u_\varepsilon) = 0 \quad (1.5\text{-}2)$$

with the NF Energy eq. (1.3-10)

$$p_N^2(t; p_I) + \frac{4}{\gamma^2} \cdot Sin^2[\frac{\gamma \cdot u_N(t; p_I)}{2}] = p_I^2 \ , \qquad (1.5\text{-}3)$$

having a family periodic NF solution exhibiting either Libration or Rotation, depending upon the initial momentum value p_I.

PERIODIC WB VDPP SOLUTIONS EXHIBITING LIBRATION

Given Initial Momentum values p_L such that $\gamma \cdot p_L \in (0,2)$, the NF Energy eq. (1.5-3) has a periodic solution $u_L(t; p_L)$ exhibiting Libration, expressible in terms of Elliptic functions [see Annex A] with a modulus value $k_L \equiv \gamma \cdot p_L / 2$, as

$$\begin{aligned} &Sin[\frac{\gamma \cdot u_L(t; p_L)}{2}] = k_L \cdot sn(t, k_L) \ , \\ &p_N(t; p_L) = p_L \cdot cn(t, k_L) \ , \end{aligned} \qquad (1.5\text{-}4)$$

with a full cycle period $T_L[p_L] = 4 \cdot \mathrm{K}(k_L)$. A particular NF solution will also be the periodic WD vdPP solution $U_0(t)$ satisfying eq. (1.5-2) when it has, for each *allowable* Softness parameter value γ, an Initial Momentum $p_*[\gamma]$ with corresponding modulus value $k_* \equiv \gamma \cdot p_*[\gamma]/2$ satisfying the Dissipation Energy constraint

$$(1.5\text{-}5) \quad \int_0^{4 \cdot \mathrm{K}(k_*)} dt \ \left[[4 \cdot \gamma^{-2} \cdot k_*^2 \cdot sn^2(t, k_*)]^d - 1 \right] \cdot [1 - sn^2(t; k_*]^{m+1} = 0 \ ,$$

imposed by eq. (1.1-14), i.e. when, for each allowable Elliptic function modulus value $k_* \in [0,1)$,

$$(1.5\text{-}6) \qquad \gamma = 2 \cdot k_* \cdot \left[\frac{\int_0^{4 \cdot K(k_*)} dt \ [sn^2(t, k_*)]^d \cdot [1\text{-}sn^2(t, k_*)]^{m+1}}{\int_0^{4 \cdot K(k_*)} dt \ [1 - sn^2(t, k_*)]^{m+1}} \right]^{\frac{1}{2d}}$$

defines a monotonically increasing *allowable* Softness parameter value $\gamma \in [0, \Gamma_{LMAX})$, with

$$\Gamma_{LMAX} \equiv 2 \cdot \left[\frac{\int_0^{4 \cdot K(1)} dt \; [sn^2(t,1)]^d \cdot [1\text{-}sn^2(t,1)]^{m+1}}{\int_0^{4 \cdot K(k_*)} dt \; [1 - sn^2(t,1)]^{m+1}} \right]^{\frac{1}{2d}} ; \tag{1.5-7}$$

it further follows from the modulus' definition that

$$p_* = \frac{2 \cdot k_*}{\gamma[k_*]} \tag{1.5-8}$$

is the necessary NF Initial Momentum. Taken together, with $k_* \in [0,1)$ as a parameter, eqs. (1.5-6) and (1.5-8) permit recovery of the particular Initial Momentum $P_0[0;\gamma,d,m] = p_*$ necessary for the Librational periodic NF solution $u_L(t;p_*)$ to be the periodic WD vdPP solution of eq. (1.5-2) exhibiting Libration.

The definition

$$A_{2n}(k) \equiv \frac{1}{4 \cdot K(k)} \cdot \int_0^{4 \cdot K(k)} dt \; sn^{2n}(t,k) \tag{1.5-9}$$

satisfying the recurrence relationship [BYRD & FRIEDMAN; 1961]

$$A_{2n+2}(k) = \frac{2n \cdot (1+k^2) \cdot A_{2n}(k) + (1-2n) \cdot A_{2n-2}(k)}{(2n+1) \cdot k^2} , \tag{1.5-10}$$

with eq. (1.5-9) supplying the initial value $A_0(k) = 1$, then permits eq. (1.5-6) to be expressed in the explicit form

$$\gamma = 2 \cdot k_* \cdot \left[\frac{\sum_{\ell=0}^{m+1} \binom{m+1}{\ell} (-)^\ell A_{2d+2\ell}(k_*)}{\sum_{\ell=0}^{m+1} \binom{m+1}{\ell} (-)^\ell A_{2\ell}(k_*)} \right]^{\frac{1}{2d}} , \tag{1.5-11}$$

with a Maximum Allowed Value $\gamma_{LMAX}[d,m]$ defined by eq. (1.5-7) having the values shown in Table 2.4-1.

TABLE 2.4-1 – REPRESENTATIVE VALUES OF $\gamma_{LMAX}[d,m]$

	$d=1$	$d=2$	$d=3$	$d=5$	$d=10$
$m=0$	**1.15470**	**1.33748**	**1.44604**	**1.57359**	**1.71759**
$m=1$	**0.89443**	**1.08216**	**1.20409**	**1.35897**	**1.55127**
$m=2$	**0.75593**	**0.93428**	**1.05582**	**1.21758**	**1.43133**
$m=3$	**0.66667**	**0.83445**	**0.95232**	**1.11420**	**1.33791**
$m=5$	**0.55470**	**0.70437**	**0.81339**	**0.96923**	**1.19818**
$m=10$	**0.41703**	**0.53752**	**0.62884**	**0.76576**	**0.98370**

In the special case $\gamma \to 0$, where

$$G[u] = \lim_{\gamma \to 0} \frac{1}{\gamma} \cdot Sin(\gamma \cdot u) \to u \ ,$$

$$\lim_{k \to 0} sn(t,k) \to Sin(t) \ ,$$

$$\lim_{k \to 0} cn(t,k) \to Cos(t) \ ,$$

eq. (1.5-8) can be re-written to express

$$P_0[0;0,d,m] = \left[\frac{\int_0^{2\pi} dt \ Cos^{2m+2}(t)}{\int_0^{2\pi} dt \ Sin^{2d}(t) \cdot Cos^{2m+2}(t)} \right]^{\frac{1}{2d}} \qquad (1.5\text{-}12)$$

as the Initial Momentum necessary for the periodic NF solution $u_L(t, p_L)$ exhibiting Libration to be the periodic WD vdPP solution of eq. (1.5-2), thereby replicating the GvdP result of eq. (1.3-5).

PERIODIC WB VDPP SOLUTIONS EXHIBITING ROTATION

Given Initial Momentum values p_R such that $\gamma \cdot p_R \in (2, \infty)$, the NF Energy eq. (1.5-3) has a periodic solution $\tilde{u}_R(t; p_R)$ exhibiting Rotation, expressible in terms of Elliptic functions [see Annex A] with a modulus value $k_R \equiv 2/\gamma \cdot p_R$ and auxiliary-time $s \equiv k_R^{-1} \cdot t$ as

$$\text{(1.5-13)} \qquad \begin{aligned} Sin[\frac{\gamma \cdot \tilde{u}_R(t; p_I)}{2}] &= sn(s, k_R) \ , \\ p_N(t; p_R) &= p_R \cdot dn(t, k_R) \ , \end{aligned}$$

with $\gamma \cdot \tilde{u}_R(t; p_I)$ increasing by 2π over a full cycle period $T_R[p_R] = 2 \cdot k_R \cdot \mathrm{K}(k_R)$. A particular NF solution will also be the periodic WD vdPP solution $U_0(t)$ satisfying eq. (1.5-2) when it has, for each *allowable* Softness parameter value γ, an Initial Momentum $p_*[\gamma]$ with corresponding modulus value $k_* \equiv 2/(\gamma \cdot p_*[\gamma])$ satisfying the Dissipation Energy constraint

$$\text{(1.5-14)} \int_0^{2 \cdot \mathrm{K}(k_*)} ds \ \left[[4 \cdot \gamma^{-2} \cdot sn^2(s, k_*)]^d - 1 \right] \cdot [1 - k_*^2 \cdot sn^2(s; k_*)]^{m+1} = 0 \ ,$$

imposed by eq. (1.1-14), i.e. when, for each allowable Elliptic function modulus value $k_* \in [0,1)$,

$$\text{(1.5-15)} \qquad \gamma \equiv \left[\frac{\int_0^{2 \cdot K(k_*)} ds \ [4 \cdot sn^2(s, k_*)]^d \cdot [1 - k_R^2 \cdot sn^2(s, k_*)]^{m+1}}{\int_0^{2 \cdot K_0(k_R)} ds \ [1 - k_*^2 \cdot sn^2(s, k_*)]^{m+1}} \right]^{\frac{1}{2d}}$$

defines a monotonically increasing *allowable* Softness parameter value $\gamma \in (\Gamma_{LMAX}, \Gamma_{RMAX}]$; it further follows from the modulus' definition that

$$p_* = \frac{2}{k_* \cdot \gamma[k_*]} \tag{1.5-16}$$

is the corresponding NF Initial Momentum. Taken together, with k_* as a parameter, eqs. (1.5-15) and (1.5-16) permit recovery of the particular Initial Momentum $P_0[0;\gamma,d,m] = p_*$ necessary for the Rotational periodic NF solution $u_{\mathrm{R}}(t;p_*)$ to be the periodic WD vdPP solution of eq. (1.5-2) exhibiting Rotation.

The definition of $\mathcal{A}_{2n}(k)$ in eq. (1.5-9) then permits eq. (1.5-15) to be expressed in the explicit form

$$\gamma = \frac{2}{k_*} \cdot \left[\frac{\sum_{\ell=0}^{m+1} \binom{m+1}{\ell} (-)^{\ell} k_*^{2d+2\ell} \cdot \mathcal{A}_{2d+2\ell}(k_*)}{\sum_{\ell=0}^{m+1} \binom{m+1}{\ell} (-)^{\ell} k_*^{2\ell} \cdot \mathcal{A}_{2\ell}(k_*)} \right]^{\frac{1}{2d}} \tag{1.5-17}$$

In the special case:

$$\begin{aligned} &\lim_{k \to 0} sn(s,k) \to Sin(s) \ , \\ &\lim_{k \to 0} cn(s,k) \to Cos(s) \ , \end{aligned} \tag{1.5-18}$$

eq. (1.5-15) can be re-written to express

$$\gamma_{RMAX} = 2 \cdot \left[\frac{1}{\pi} \cdot \int_0^{\pi} ds \ Sin^{2d}(s) \right]^{\frac{1}{2d}} = \left[\frac{(2d)!}{(d!)^2} \right]^{\frac{1}{2d}} \tag{1.5-19}$$

as the Maximum Allowable Softness parameter value permitting the periodic NF solution $u_L(t,p_{\mathrm{R}})$ exhibiting Rotation to be the periodic WD vdPP solution of eq. (1.5-2), with the representative values (independent of the Momentum exponent value m) shown in Table 2.4-2.

TABLE 2.4-2 – REPRESENTATIVE VALUES OF $\Gamma_{RMAX}[d,m]$

	$d=1$	$d=2$	$d=3$	$d=5$	$d=10$
$m \in [1,\infty]$	**1.41412**	**1.56508**	**1.64755**	**1.73836**	**1.83371**

SUMMARY OF PERIODIC WB VDPP SOLUTIONS

Figure 2.4-3 shows the periodic WD vdPP solutions defined by eqs. (1.5-4) and (1.5-13) to make a seamless transition between Libration and Rotation because the Softness parameter values defined by eqs. (1.5-11) and (1.5-17) make a seamless transition as the elliptic modulus $k_* \to 1$. Curves to the left of the dot-dash separatrix show values yielding a periodic WD vdPP solution exhibiting Libration while curves to the right of the dot-dash separatrix show values yielding a periodic WD vdPP solution exhibiting Rotation. For completeness, Figure 2.4-4 shows the corresponding Maximum Displacement $U_0[\gamma;d,m]$ as a function of the overall Softness parameter γ for representative Dissipation Exponent values d and m when the periodic vdPP solution exhibits Libration.

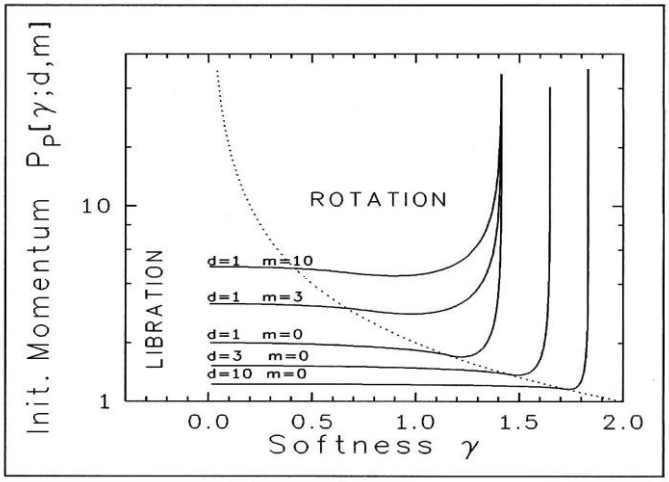


FIGURE 2.4-3 – REPRESENTATIVE VALUES OF THE INITIAL MOMENTUM $P_P[\gamma;d,m]$ NECESSARY FOR THE VDPP TO HAVE A PERIODIC W-D SOLUTION

Solutions exhibiting Rotation will have a complete Period with the momentum displaying only a positive full cycle. For this reason, it is useful to define the overall WD vdPP Period $T_P[\gamma;d,m]$ as that of the momentum's positive half cycle when its periodic motion exhibits Libration; upon using this definition, Fig. 2.4-4 shows the WD vdPP Librational Period increases monotonically from an initial value $T_P[0;d,m]=\pi$ to a terminal value $T_P[\gamma_{LM};d,m]=\infty$ while the WD vdPP Rotational Period decreases monotonically from an initial value $T_P[\gamma_{LM};d,m]=\infty$ to a terminal value $T_P[\gamma_{MR};d,m]=0$.

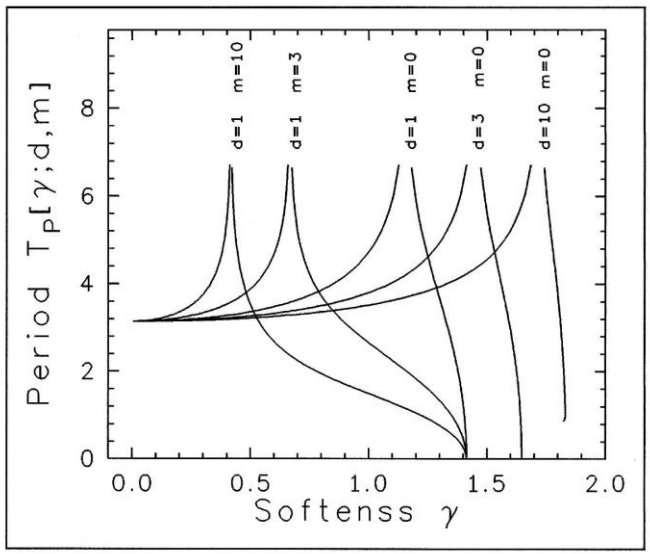


FIGURE 2.4-4 – REPRESENTATIVE VALUES OF THE MAXIMUM LIBRATIONAL DISPLACEMENT $U_P[\gamma;d,m]$ OF A PERIODIC W-D SOLUTION

Tables 2.4-3 and 2.4-4 [see Annex T] provide representative values of $P_1[\gamma;d,m]$ necessary for a Periodic WD vdPP solution.

Chapter 2: The Warped-Time Method [WTM]

Equation Chapter (Next) Section 1

2.1 INTRODUCTION TO THE WTM

The previous Chapter showed how a particular member in the family of periodic NF solutions $u_N(t;p_I)$ satisfying

$$\ddot{u}_N(t;p_I)+G[u_N(t;p_I)]=0\ , \tag{2.1-1}$$

with period $T_N[p_I]$ was also a periodic Weak-Dissipation solution

$$U_0(t)=u_N(t;p_*)\ , \tag{2.1-2}$$

of the XvdP equation

$$\ddot{u}_\varepsilon(t)+\varepsilon\cdot F\left[u_\varepsilon(t);p_\varepsilon(t)\right]\cdot p_\varepsilon(t)+G[u_\varepsilon(t)]=0 \tag{2.1-3}$$

when there existed a particular initial NF parameter p_* yielding

$$\int_0^{T_N[p_*]}dt\ F[u_N(t;p_*);p_N(t;p_*)]\cdot p_N^2(t;p_*)=0\ . \tag{2.1-4}$$

This Chapter introduces a Warped-Time Method (WTM) for expanding the WD result shown of eq. (2.1-2) to all values of the nonlinearity parameter ε. As its first step, the WTM expressed NF equation (2.1-1) in terms of an auxiliary-time τ as

$$u''_N(\tau;p_I)+G[u_N(\tau;p_I)]=0\ ; \tag{2.1-5}$$

it then assures the existence of a Restoring Force $G[u]$ permitting the existence of a periodic solution $u_N(\tau;p_I)$ with period $T_N[p_I]$and imposed the initial conditions

$$\begin{aligned} u_N(0;p_I)&=0\ ,\\ p_N(0;p_I)&=p_I\ . \end{aligned} \tag{2.1-6}$$

The WTM continues by expressing elapsed-time t as a particular *warped* version

(2.1-7) $$t \equiv \int_0^{\tau} \frac{ds}{h_\varepsilon(s)}$$

of an auxiliary-time τ, with the constraint

(2.1-8) $$h_\varepsilon(\tau) \equiv \frac{d\tau}{dt} > 0$$

assuring a one-to-one relationship between elapsed-time t and auxiliary-time τ. In analogy with eq. (2.1-2) , the WTM then expresses the periodic XvdP solution of eq. (2.1-3) as

(2.1-9) $$U_\varepsilon(t) = u_N(\tau; p_I) \ ,$$

with the function $h_\varepsilon(\tau)$ and the appropriate initial NF momentum to be determined.

The WTM determines the function $h_\varepsilon(\tau)$ by differentiating eq. (2.1-9) with respect to elapsed-time and eq. (2.1-8) to obtain

(2.1-10) $$\dot{U}_\varepsilon(t) = \dot{u}_N(\tau; p_I) = p_N(\tau; p_I) \cdot \frac{d\tau}{dt} = h_\varepsilon(\tau) \cdot p_N(\tau; p_I) \ ;$$

it similarly differentiates eq. **Error! Reference source not found.**(2.1-10) with respect to elapsed-time t and again uses eq. (2.1-8) to obtain

(2.1-11) $$\begin{aligned} \ddot{U}_\varepsilon(t) &= h_\varepsilon(\tau) \cdot \dot{p}_N(\tau; p_I) + \dot{h}_\varepsilon(\tau) \cdot p_N(\tau; p_I) = \\ &= [h_\varepsilon(\tau) \cdot p'_N(\tau; p_I) + h'_\varepsilon(\tau) \cdot p_N(\tau; p_I)] \cdot \frac{d\tau}{dt} = \\ &= h_\varepsilon^2(\tau) \cdot p'_N(\tau; p_I) + h_\varepsilon(\tau) \cdot h'_\varepsilon(\tau) \cdot p_N(\tau; p_I) \ . \end{aligned}$$

The WTM now substitutes the representations of eqs. **Error! Reference source not found.**(2.1-9), (2.1-10), and (2.1-11) into eq. (2.1-3) and recovers

(2.1-12) $$\begin{aligned} & h_\varepsilon^2(\tau) \cdot p'_N(\tau; p_I) + h_\varepsilon(\tau) \cdot h'_\varepsilon(\tau) \cdot p_N(\tau; p_I) + G[u_N(\tau; p_I)] = \\ & = -\varepsilon \cdot F\left[u_N(\tau; p_I); h_\varepsilon(t) \cdot p_N(\tau; p_I)\right] \cdot h_\varepsilon(t) \cdot p_N(\tau; p_I) \ ; \end{aligned}$$

it further uses the alternative form

$$G[u_N(\tau;p_I)] = -u''_{\mathbf{N}}(\tau;p_I) \equiv -p'_N(\tau;p_I)\ , \tag{2.1-13}$$

of eq. (2.1-5) to express eq. (2.1-12) as

$$\begin{aligned}&[h_\varepsilon^2(\tau)\text{-}1]\cdot p'_N(\tau;p_I)+h_\varepsilon(\tau)\cdot h'_\varepsilon(\tau)\cdot p_N(\tau;p_I)=\\&=-\varepsilon\cdot F\left[u_N(\tau;p_I);h_\varepsilon(t)\cdot p_N(\tau;p_I)\right]\cdot h_\varepsilon(t)\cdot p_N(\tau;p_I)\ .\end{aligned} \tag{2.1-14}$$

As its penultimate step, the WTM introduces the *Warping Function*

$$W_\varepsilon(\tau;p_I) \equiv \frac{h_\varepsilon^2(\tau)-1}{\varepsilon}\cdot p_N^2(\tau;p_I)\ , \tag{2.1-15}$$

expresses eq. (2.1-14)**Error! Reference source not found.** in its preferred form

$$\begin{aligned}&W'_\varepsilon(\tau;p_I) = -2\cdot p_N^2(\tau;p_I)\cdot\\&\cdot F\left[u_N(\tau;p_I);h_\varepsilon(\tau)\cdot p_N(\tau;p_I)\right]\cdot h_\varepsilon(\tau)\ ,\\&h_\varepsilon(\tau) \equiv \sqrt{1+\varepsilon\cdot\frac{W_\varepsilon(\tau;p_I)}{p_N^2(\tau;p_I)}}\ ,\end{aligned} \tag{2.1-16}$$

and uses the boundary condition

$$W_\varepsilon(\tau_M;p_I) = W_M \tag{2.1-17}$$

for some particular auxiliary-time τ_M to obtain the Warping Function $W_\varepsilon(\tau;p_I)$ and the time-warping function $h_\varepsilon(\tau)$.

The WTM completes its recovery of the periodic XvdP solution defined by eq. (2.1-9) through recovering the particular NF initial momentum value $p_I = p_{**}$ ensuring the Warping Function satisfies the periodicity constraint

$$W_\varepsilon(\tau+T_N[p_{**}];p_{**}) = W_\varepsilon(\tau;p_{**}) \tag{2.1-18}$$

implicit in its definition by eq. (2.1-15); it also uses eqs. (2.1-7) and (2.1-16) to express the XvdP Initial Momentum $P_1[\varepsilon]$ Period $T_0[\varepsilon]$ as

$$P_0[\varepsilon] = h_\varepsilon(0)\cdot p_{**} = \sqrt{p_{**}^2 + \varepsilon\cdot W_\varepsilon(0;p_{**})}\ ,$$
$$T_0[\varepsilon] \square \int_0^{T_N[p_{**}]} \frac{ds}{h_\varepsilon(s)} = \int_0^{T_N[p_{**}]} ds \frac{\left|p_N(s;p_{**})\right|}{\sqrt{p_N^2(s;p_{**}) + \varepsilon\cdot W_\varepsilon(s,p_{**})}}\ . \qquad (2.1\text{-}19)$$

A WEAK DISSIPATION WTM RESULT

Under Weak Dissipation conditions, where the nonlinearity parameter value $\varepsilon \to 0^+$, the Warping Function defined by eq. (2.1-16) assumes the simple form

(2.1-20) $$W_\varepsilon(\tau;p_{**}) = -2\cdot\int_0^\tau ds\ p_N^2(s;p_{**})\cdot F\left[u_N(s;p_{**});p_N(s;p_{**})\right]\ ,$$

with the periodicity condition of eq. (2.1-18) becoming

(2.1-21) $$\int_0^{T_N[p_{**}]} ds\ p_N^2(s;p_{**})\cdot F\left[u_N(s;p_{**});p_N(s;p_{**})\right] = 0\ ,$$

thereby replicating the WD solution requirement of eq. (2.1-4).

2.2 WTM ANALYSIS OF THE LVDP EQUATION

The WTM recovers the periodic solutions of the LvdP equation

(2.1-22) $$\ddot{u}(t) + \varepsilon\left[u^2(t) - 1\right]\dot{u}(t) + u(t) = 0$$

by first observing that, with a linear Restoring Force $G_N(u) = u$, the NF eq. (2.1-1) has a periodic solution

(2.1-23) $$u_N(\tau;p_I) = p_I\cdot Sin(\tau)\ ,$$
$$p_N(\tau;p_I) = p_I\cdot Cos(\tau)\ ,$$

exhibiting Libration with the (fixed) period $T[p_I] = 2\pi$ and satisfying the initial conditions of eq. (2.1-6)**Error! Reference source not found.**; thus, it can express eq. **Error! Reference source not found.**(2.1-16) as

$$W'_\varepsilon(\tau;p_I) = -2\cdot p_I \cdot \left|Cos(\tau)\right| \cdot \left[p_I^2 \cdot Sin^2(\tau) - 1\right] \cdot \\ \cdot \sqrt{p_I^2 \cdot Cos^2(\tau) + \varepsilon \cdot W_\varepsilon(\tau;p_I)} \ , \tag{2.1-24}$$

with the NF initial momentum p_I chosen to satisfy the periodicity condition of eq. (2.1-18), i.e.

$$W_\varepsilon(\tau + 2\pi; p_I) = W_\varepsilon(\tau; p_I) \ . \tag{2.1-25}$$

The form of eq. (2.1-24), however, **Error! Reference source not found.** remains unchanged upon substituting the *abbreviated* periodicity condition

$$W_\varepsilon(\tau + \pi; p_I) = W_\varepsilon(\tau; p_I) \tag{2.1-26}$$

because the periodic NF solution $u_N(\tau;p_I)$ has symmetrical odd and even cycles with the same time-warping.

The WTM next recovers the boundary condition of eq. (2.1-17) by setting $\tau_M = -\pi/2$ so that $p_N(\tau_M;p_I) = 0$, with eq. (2.1-17) yielding

$$W_\varepsilon(\tau_M; p_I) = 0 \ ; \tag{2.1-27}$$

it also uses an *offset-time* $s \equiv \tau - \tau_M$ to express eq. (2.1-24) in the simplified representation

$$\bar{W}'_\varepsilon(s;p_I) = -2\cdot p_I \cdot Sin(s) \cdot \left[p_I^2 \cdot Cos^2(s) - 1\right] \cdot \\ \cdot \sqrt{p_I^2 \cdot Sin^2(s) + \varepsilon \cdot \bar{W}_\varepsilon(s;p_I)} \ , \tag{2.1-28}$$

satisfying the *abbreviated* periodicity condition

$$\bar{W}_\varepsilon(\pi;p_I) = \bar{W}_\varepsilon(s;p_I) = 0 \tag{2.1-29}$$

over the offset-time's Basic Interval $s \in [0,\pi)$.

The terms within eq. (2.1-28) permit a qualitative description of its behavior first showing that when $p_I^2 \le 1$, $\bar{W}'_\varepsilon(s;p_I)$ will retain a non-negative value throughout the Basic Interval $s \in [0,\pi)$, precluding the possibility of $\bar{W}_\varepsilon(\pi;p_I) = 0$ required by the abbreviated periodicity condition eq.**Error! Reference source not found.** (2.1-29); conversely, with $p_I^2 > 1$,

eq. (2.1-28)**Error! Reference source not found.** shows the LvdP's offset-time Warping Function $\bar{W}_\varepsilon(s)$ to have two points of inflexion

$$\begin{aligned} s_1 &\approx -Cos^{-1}(p_I^{-1}) \,, \\ s_2 &\approx +Cos^{-1}(p_I^{-1}) \,, \end{aligned} \tag{2.1-30}$$

within the Basic Interval $s \in [0, \pi)$. The constraint $p_I^2 > 1$ further implies that with $s^2 \ll 1$,

$$\bar{W}_\varepsilon(s; p_I) \approx -\frac{2 \cdot p_I^2 \cdot [p_I^2 - 1]}{3} s^3 \,. \tag{2.1-31}$$

Equations **Error! Reference source not found.** and (2.1-31) then show the offset-time Warping Function $\bar{W}_\varepsilon(s)$ to contain three contiguous Regions:

Region $\Re_I$: Within the *Initial* interval $s \in (0, s_1)$, the LvdP Warping Function $\bar{W}_\varepsilon(s; p_I)$ continually *decreases* from its *initial* null value as offset-time s increases, until attaining its minimum value $W_{MIN} \approx \bar{W}_\varepsilon(s_1; p_I) < 0$.

Region $\Re_X$: Within the *Intermediate* interval $s \in (s_1, s_2)$, the LvdP Warping Function $\bar{W}_\varepsilon(s; p_I)$ continually increases from its minimum value $W_{MIN} < 0$ as the offset-time s increases, until attaining its maximum value $W_{MAX} \approx \bar{W}_\varepsilon(s_2; p_I) > 0$; it also attains its *intermediate* null value $\bar{W}_\varepsilon(s_N; p_{**}) = 0$ at an offset-time $s_1 \le s_N \le s_2$.

Region $\Re_T$: Within the *Terminal* interval $s \in (s_2, \pi)$, the LvdP Warping Function $\bar{W}_\varepsilon(s; p_I)$ continually *decreases* from its maximum value $W_{MAX} > 0$ as the auxiliary-time s increases, until attaining its terminal null value $\bar{W}_\varepsilon(\pi; p_{**}) = 0$.

WEAK-DISSIPATION BEHAVIOR

Under the Weak-Dissipation condition $\varepsilon \to 0^+$, eq. (2.1-28) **Error! Reference source not found.**has the immediate solution

$$\bar{W}_0(s;p_I) = -2p_I^2 \cdot \int_0^s dx\ Sin^2(x) \cdot \left[p_I^2 \cdot Cos^2(x) - 1 \right] =$$

(2.1-32)

$$= -2p_I^2 \cdot \left[\frac{p_I^2 - 4}{8} s + \frac{1}{4} Sin(2s) - \frac{p_I^2}{32} Sin(4s) \right],$$

from which it follows that the NF initial momentum must have the particular value $p_{**}^2 = 4$ to ensure $\bar{W}_\varepsilon(\pi;p_I) = 0$ as required by eq.(2.1-29) with the simple result $W_0(s) = -8 \cdot Sin^3(s) \cdot Cos(s)$ shown in Fig. 3.2-1.

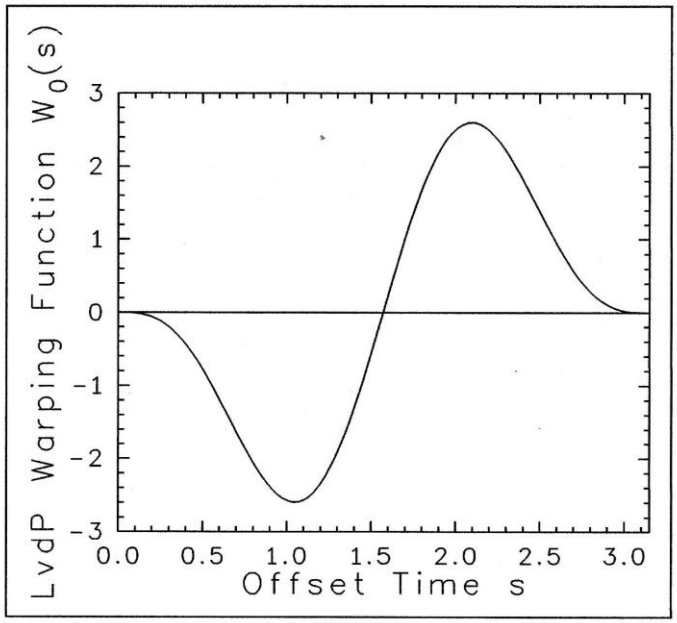


FIGURE 3.2-1 – WEAK-DISSIPATION LVDP OFFSET WARPING FUNCTION $W_0(s) \equiv \lim_{\varepsilon \to 0^+} W_\varepsilon(s)$

STRONG-DISSIPATION BEHAVIOR

Under the Strong-Dissipation condition $\varepsilon \to \infty$, substitution of the *asymptotic* Warping Function

$$\bar{W}_\varepsilon(s;p_I) = \varepsilon \cdot \bar{B}(s;p_I) \tag{2.1-33}$$

into eq. (2.1-28) **Error! Reference source not found.**yields

$$\overline{B}'(s) = -2 \cdot p_I \cdot Sin(s) \cdot \left[p_I^2 \cdot Cos^2(s) - 1 \right] \cdot \sqrt{\frac{p_I^2 \cdot Sin^2(s)}{\varepsilon^2} + \overline{B}(s)} \ . \quad (2.1\text{-}34)$$

throughout the Basic Interval $s \in [0, \pi)$. Thus, within the Initial Region $\Re_I$, where $\overline{B}(0) = 0$ and $\overline{B}'(s) \le 0$, eq. **Error! Reference source not found.** imposes the constraint

$$-\frac{p_I^2 Sin^2(s)}{\varepsilon^2} \le \overline{B}(s) \le 0 \ , \quad (2.1\text{-}35)$$

with its asymptotic form

$$(2.1\text{-}36) \qquad -\lim_{\varepsilon \to \infty} \frac{p_I^2 Sin^2(s)}{\varepsilon^2} = 0^- \le B(s) \le 0$$

leading to the asymptotic Region $\Re_I$ result $\overline{B}(s) = 0$, with the particular result $\overline{B}(s_1) = 0$ further requiring $s_0 \equiv s_1$.

Within the Intermediate and Terminal Regions $\Re_X$ & $\Re_T$, where

$$(2.1\text{-}37) \qquad \begin{aligned} &\tilde{B}(s) \ge 0 \ , \\ &\lim_{\varepsilon \to \infty} \sqrt{\frac{p_I^2 \cdot Sin^2(s)}{\varepsilon^2} + \overline{B}(s)} = \sqrt{\overline{B}(s)} \ge 0 \ , \end{aligned}$$

eq. **Error! Reference source not found.** assumes the asymptotic form

$$(2.1\text{-}38) \qquad \overline{B}'(s) = -2 \cdot p_I \cdot Sin(s) \cdot \left[p_I^2 \cdot Cos^2(s) - 1 \right] \cdot \sqrt{\overline{B}(s)} \ ,$$

with the immediate solution

$$(2.1\text{-}39) \qquad \sqrt{\overline{B}(s)} = \frac{p_I^3 [Cos^3(s) + 1]}{3} - p_I [Cos(s) + 1] \ ,$$

from which it follows that the NF initial momentum must have the particular value p_{**} satisfying the cubic equation

$$(2.1\text{-}40) \qquad \frac{p_{**}^3 [Cos^3(s_1) + 1]}{3} - p_{**} [Cos(s_1) + 1] = 0 \ .$$

The definition $p_{**}Cos(s_1) = 1$ supplied by eq.
Error! Reference source not found. then allows eq.
Error! Reference source not found. to be expressed in the definitive form

$$\frac{p_{**}^3}{3} - p_{**} - \frac{2}{3} = 0 \ , \tag{2.1-41}$$

yielding $p_{**} = 2$ and permitting eq. (2.1-39) to be expressed within Regions $\Re_X$ & $\Re_T$ as

$$\sqrt{\bar{B}(s)} = \frac{2}{3} \cdot \left[4 \cdot Cos^3(s) - 3 \cdot Cos(s) + 1 \right] \ . \tag{2.1-42}$$

Taken together, the results of eqs. (2.1-36) and (2.1-42) yield the complete behavior of $\bar{B}(s)$ over the Fundamental Interval $s \in [0, \pi)$ shown in Fig. 3.2-2 .

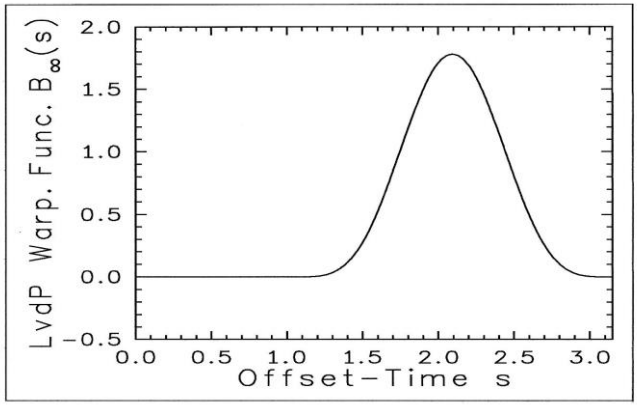


FIGURE 3.2-2 – STRONG DISSIPATION LVDP OFFSET WARPING FUNCTION $\bar{B}(s) \equiv \lim_{\varepsilon \to \infty} \varepsilon^{-1} \cdot \bar{W}_\varepsilon(s; p_I)$

Equation (2.1-19) provides the LvdP's *asymptotic normalized* Initial Momentum and Period as

$$P_1[\infty] \equiv \lim_{\varepsilon\to\infty} \frac{P_0[\varepsilon]}{\varepsilon} = \lim_{\varepsilon\to\infty} \sqrt{\frac{p_{**}^2}{\varepsilon^2} + \bar{B}(\pi/2)} = \sqrt{\bar{B}(\pi/2)} = \frac{2}{3} \ ,$$
$$T_1[\infty] \equiv \lim_{\varepsilon\to\infty} \frac{T_0[\varepsilon]}{\varepsilon} = \lim_{\varepsilon\to\infty} \frac{2}{\varepsilon} \cdot \int_0^{\pi} dx \ \frac{2 \cdot Sin(x)}{\sqrt{4 \cdot Sin^2(s) + \varepsilon \cdot \sqrt{\bar{B}(s)}}} \ . \qquad (2.1\text{-}43)$$

The second line of eq. **Error! Reference source not found.** shows the LvdP's asymptotic normalized Period receives no contribution from Intermediate and Terminal Regions $\Re_X$ and $\Re_T$ where $\varepsilon \cdot \sqrt{\bar{B}(s)}$ is unbounded; there is, however, a finite contribution to the asymptotic normalized period from the Initial Region $\Re_I$. In particular, upon setting

(2.1-44) $$W_\varepsilon(s) = \frac{e_\varepsilon^2(s)}{\varepsilon^3} - \frac{4 \cdot Sin^2(s)}{\varepsilon} \ .$$

and substituting the Warping Function definition of eq. **Error! Reference source not found.** into eq.(2.1-28), it follows that

(2.1-45) $$2 \cdot \frac{\tilde{e}_\varepsilon(s) \cdot \tilde{e}'_\varepsilon(s)}{\varepsilon^3} - 8 \cdot \frac{Sin(s) \cdot Cos(s)}{\varepsilon} = \\ = -4 \cdot Sin(s) \cdot \left[4 \cdot Cos^2(s) - 1\right] \cdot \frac{\tilde{e}_\varepsilon(s)}{\varepsilon} \ ;$$

upon then equating terms containing ε^{-1}, the result

(2.1-46) $$\tilde{e}_\varepsilon(s) = \frac{2 \cdot Cos(s)}{4 \cdot Cos^2(s) - 1}$$

leads eq. **Error! Reference source not found.** to provide the asymptotic result

(2.1-47) $$T_1[\varepsilon] \equiv 2 \cdot \int_0^{Cos^{-1}(1/2)} dx \ \frac{[4 \cdot Cos^2(x) - 1] \cdot Sin(x)}{Cos(x)} = 3 - 2 \cdot Ln[2] \ ,$$

first recovered by van der Pol and shown in eq. **A.1-7)**.

INTERMEDIATE-DISSIPATION BEHAVIOR

For intermediate nonlinearity parameter values $\varepsilon \in (1,100)$, numerical integrations provide the LvdP Period $T_0[\varepsilon]$, Initial Momentum $P_0[\varepsilon]$ and Maximum Displacement $U_0[\varepsilon]$ for representative intermediate values of the nonlinearity parameter ε given in Table 3.2-1, with bold entries match those given in Table 1.7-1 within 0.01%; Figures 3.2-4 through 3.2-6 display the Warping Function, Elapsed Time, and Maximum Displacement for a more limited set of nonlinearity parameter values.

Table 3.2-1 – LvdP Period $T_0[\varepsilon]$, Initial Momentum $P_0[\varepsilon]$ and Maximum Displacement $U_0[\varepsilon]$ as a function of the nonlinearity parameter ε

E	$T_0[\varepsilon]$	$P_0[\varepsilon]$	$U_0[\varepsilon]$
E→0⁺	**6.28319**	**2.00000**	**2.00000**
1	**6.66329**	**2.17271**	**2.00862**
2	**7.62987**	**2.61497**	**2.01989**
3	**8.85909**	**3.16872**	**2.02330**
4	**10.20352**	**3.76234**	**2.02296**
5	**11.71223**	**4.37523**	**2.02151**
6	**13.06187**	**4.99944**	**2.01983**
7	**14.53974**	**5.63109**	**2.01822**
8	**16.03817**	**6.26802**	**2.01675**
9	**17.55212**	**6.90889**	**2.01544**
10	**19.07837**	**7.55281**	**2.01429**
20	**34.68232**	**14.07754**	**2.00779**
30	**50.54368**	**20.66771**	**2.00516**
40	**66.50136**	**27.28328**	**2.00379**

50	**82.50833**	**33.91222**	**2.00296**
60	**98.54478**	**40.54931**	**2.00240**
70	**114.60066**	**47.19188**	**2.00201**
80	**130.67020**	**53.83833**	**2.00172**
90	**146.74978**	**60.48767**	**2.00150**
100	**162.83707**	**67.13925**	**2.00132**
E→∞	***1.61371****	***6.66666****	**2.00000**

* Asymptotic value divided by ε

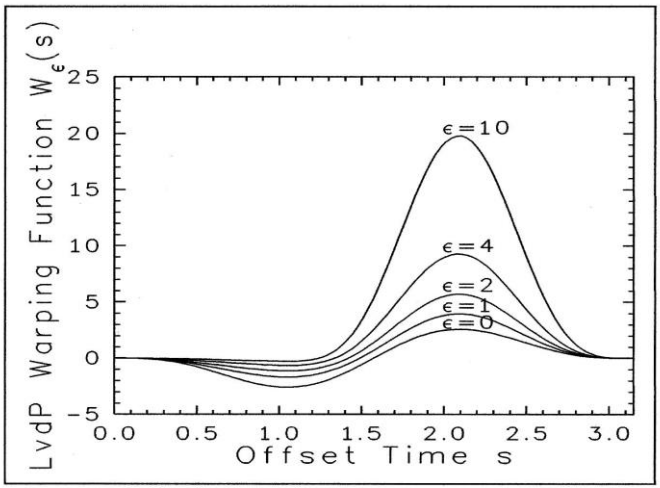


FIGURE 3.2-4 - LVDP WARPING FUNCTION $W_\varepsilon(s)$ AS A FUNCTION OF OFFSET-TIME s

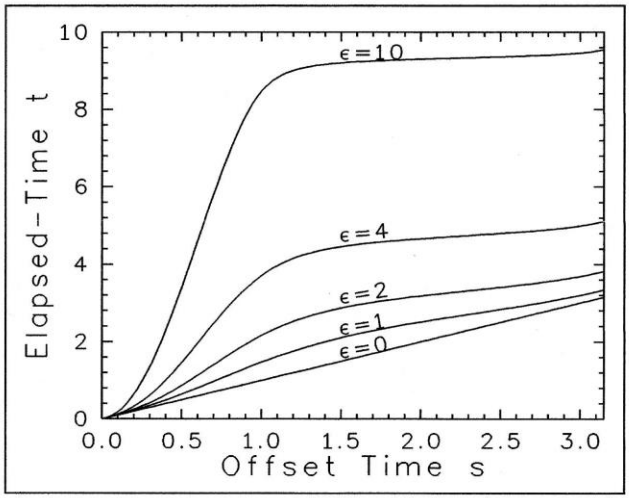


FIGURE 3.2-5 - LVDP ELAPSED-TIME t AS A FUNCTION OF OFFSET-TIME s

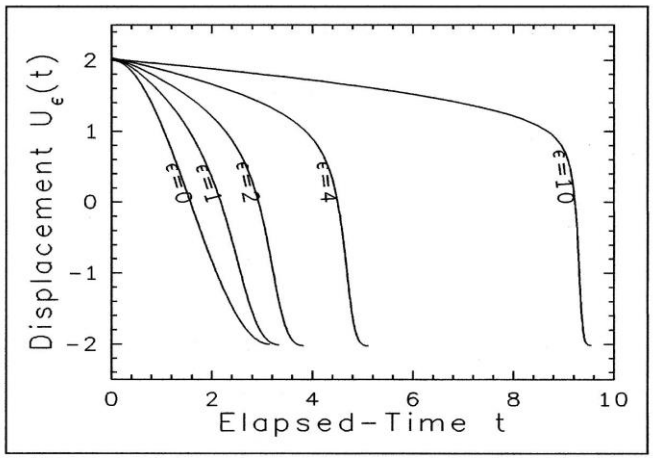


FIGURE 3.2-6 - LVDP DISPLACEMENT $U_\varepsilon(t)$ AS A FUNCTION OF ELAPSED-TIME t

Chapter 3: WTM Analysis for the GvdP

Equation Chapter (Next) Section 1

3.1 Introduction to WTM for the GvdP

In this Chapter, we use the WTM process illustrated in Sec. 3.2 to recover relationships among the Maximum Displacement $U_0[\varepsilon;\delta,\mu]$, nonlinearity parameter $\varepsilon > 0$, and the Dissipation Factor exponents $\delta > 0$, $\mu \geq 0$ necessary for the Generalized van der Pol (GvdP) equation, introduced in Sec. 2.2 as

$$\ddot{u} + \varepsilon\left[[u^2]^\delta - 1\right][\dot{u}^2]^\mu \cdot \dot{u} + u = 0 \,, \tag{3.1-1}$$

to have a periodic solution $U_\varepsilon(t)$ exhibiting Libration. Unlike previous analyses which recovered the Initial Weak Maximum Displacement $U_0[0;\delta,\mu] \equiv \lim_{\varepsilon\to 0^+} U_0[\varepsilon;\delta,\mu]$ associated with different (integer) Dissipation Factor exponent values, we now use the terminology of Sec. 2.1 to examine the (real) Dissipation Factor exponent values $\delta > 0$ and $\mu \geq 0$, together with the nonlinearity parameter value ε and the Maximum Displacement $r_0 \equiv U_0[\varepsilon;\delta,\mu]$, as *equally important elements* in characterizing a periodic solution $U_\varepsilon(t)$ of eq. (3.1-1) exhibiting Libration.

We are particularly interested in the relationship between Dissipation Factor exponent values $\delta > 0$ and $\mu \geq 0$ yielding the same Maximum Displacement r_0 for a particular nonlinearity parameter value ε because these relationships supply insight into previously unrecognized EvdP characteristics, e.g. the existence of a Critical Initial Displacement value $r_C[\varepsilon]$ such that with $r < r_C[\varepsilon]$, there is a Minimum Allowable Displacement exponent $\delta_{MIN}[\varepsilon]$, while with $r_0 > r_C[\varepsilon]$ there is a Minimum Allowable Momentum exponent $\mu_{MIN}[\varepsilon]$; only when $r_0 = r_C[\varepsilon]$ can the Dissipation Factor exponents δ and μ approach their respective minimum values simultaneously. More importantly, in contradiction of the Weak-Dissipation result showing the existence of an Initial Momentum value r_0 for all Dissipation Factor exponent values $\delta > 0$ and $\mu \geq 0$, these contours of constant Initial Momentum values r_0 show the allowable Dissipation Factor Momentum exponent values to have an asymptotic ($\varepsilon \to \infty$) maximum

possible Momentum exponent value $\mu \le 0.5$ for all Displacement exponent values $\delta \in (0,\infty]$.

We begin our investigation of interaction among the various EvdP parameter values by using the definitions of Sec. 1.2 in recovering the family of Librational solutions

$$\begin{aligned} u_N(\tau;r_0) &= r_0 \cdot Cos(\tau) \\ p_N(\tau;r_0) &\equiv u'_N(\tau;r_0) = -r_0 \cdot Sin(\tau) \end{aligned} \tag{3.1-2}$$

belonging with the NF version

$$u''_N(\tau;r_0) + u_N(\tau;r_0) = 0 \tag{3.1-3}$$

of eq. (3.1-1) and subject to the initial conditions

$$\begin{aligned} u_N(0;r_0) &= r_0 > 0, \\ p_N(0;r_0) &= 0 \ ; \end{aligned} \tag{3.1-4}$$

we then express eq. **Error! Reference source not found.** explicitly as

$$\begin{aligned} &W'_\varepsilon(\tau) = 2 \cdot r_0 \cdot Sin(\tau) \cdot \left[[r_0^2 \cdot Cos^2(\tau)]^\delta - 1 \right] \cdot [\nu_\varepsilon^2(\tau)]^\mu \cdot \nu_\varepsilon(\tau) = 0, \\ &\nu_\varepsilon(\tau) = \mathrm{sgn}[r_0 \cdot Sin(\tau)] \cdot \sqrt{r_0^2 \cdot Sin^2(\tau) + \varepsilon \cdot W_\varepsilon(\tau)} \ . \end{aligned} \tag{3.1-5}$$

It is further clear from its definition in eq. (3.1-5) that we need recover the Warping Function $W_\varepsilon(\tau)$ only over the Fundamental Interval $\tau \in [0,\pi]$, subject to the (modified) periodicity condition

$$W_\varepsilon(\pi) = W_\varepsilon(0) = 0 \ . \tag{3.1-6}$$

3.2 EVDP WEAK DISSIPATION BEHAVIOR

Under Weak Dissipation conditions ($\varepsilon \to 0^+$), we can express eq. (3.1-5) in the simplified form

$$\begin{aligned} &\lim_{\varepsilon \to 0^+} \nu_\varepsilon(s) = -r_0 \cdot Sin(\tau) \ , \\ &\lim_{\varepsilon \to 0^+} W'_\varepsilon(\tau) = 2 \cdot \left[[r_0^2 \cdot Cos^2(\tau)]^\delta - 1 \right] \cdot [r_0^2 \cdot Sin^2(\tau)]^{\mu+1} \ . \end{aligned} \tag{3.1-7}$$

The (modified) periodicity condition of eq. (3.1-6) then simply requires the NF's WD Maximum Displacement $r_{00} \equiv U_0[0;\delta,\mu]$, together with the Dissipation exponent values δ and μ, to be related to one another such that

$$\int_0^\pi dx \left[[r_{00}^2 \cdot Cos^2(x)]^\delta - 1 \right] \cdot [Sin^2(x)]^{\mu+1} = 0 \ , \tag{3.1-8}$$

i.e. we require a NF WD Maximum Displacement value

$$r_{00} = \left[\frac{\int_0^\pi dx \ [Sin^2(x)]^\mu}{\int_0^\pi dx \ [Cos^2(x)]^\delta \cdot [Sin^2(x)]^\mu} \right]^{\frac{1}{2\cdot\delta}} \ . \tag{3.1-9}$$

The Beta function definition

$$\begin{aligned} B[a,b] &\equiv 2 \cdot \int_0^{\pi/2} ds [Sin(s)]^{2a-1} \cdot [Cos(s)]^{2b-1} = \\ &= \Gamma(a) \cdot \Gamma(b) / \Gamma(a+b) \ , \end{aligned} \tag{3.1-10}$$

together with the Gamma function duplication formula

$$2 \cdot \Gamma(1/2) \cdot \Gamma(2z) = 2^{2z} \cdot \Gamma(z) \cdot \Gamma(z + 1/2) \ , \tag{3.1-11}$$

then allow eq. (3.1-9) to be expressed in the more useful form

$$r_{00} = 2 \cdot \left[\frac{\Gamma(\delta+\mu+2)}{\Gamma(\mu+2)} \cdot \frac{\Gamma(\delta+1)}{\Gamma(2\cdot\delta+1)} \right]^{\frac{1}{2\cdot\delta}} \tag{3.1-12}$$

illustrated in Fig. 4.2-1.

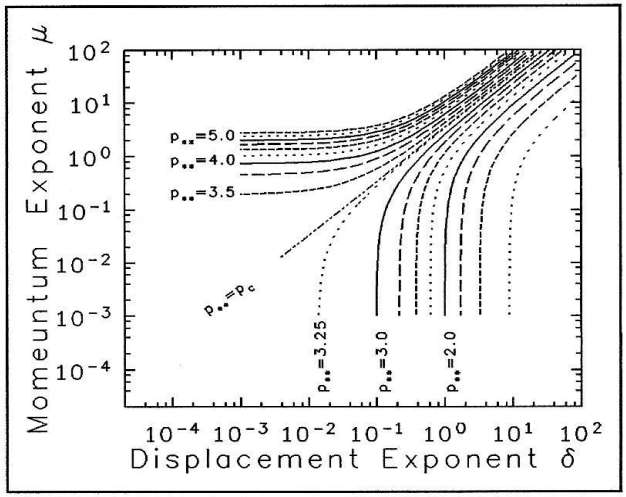


FIGURE 4.2-1 – WEAK DISSIPATION ($\varepsilon \to 0^+$) NF WD MAXIMUM DISPLACEMENT r_{00} AS A FUNCTION OF DISSIPATION FACTOR EXPONENTS δ AND μ

Figure 4.2-1. shows, in particular, the existence of a Critical Maximum Displacement value $r_C[0]$, such that every WD Maximum Displacement value $r_{00} < r_C[0]$ has an associated Critical Displacement Exponent value δ equal to or exceeding a value δ_0 satisfying

$$r_{00} = 2 \cdot \left[\frac{\Gamma(\delta_0 + 2) \cdot \Gamma(\delta_0 + 1)}{\Gamma(2 \cdot \delta_0 + 1)} \right]^{\frac{1}{2 \cdot \delta_0}} \quad ; \tag{3.1-13}$$

conversely, every WD Maximum Displacement value $r_{00} > r_C[0]$, has an associated Critical Momentum Exponent value μ equal to or exceed a value μ_0 satisfying

$$r_{00} = 2 \cdot \lim_{\delta \to 0^+} \left[\frac{\Gamma(\delta + \mu_0 + 2)}{\Gamma(\mu_0 + 2)} \cdot \frac{\Gamma(\delta + 1)}{\Gamma(2 \cdot \delta + 1)} \right]^{\frac{1}{2 \cdot \delta}} \tag{3.1-14}$$

We obtain an explicit expression for the μ_0 value satisfying (3.1-14) by using the Psi function [ABRAMOWITZ & STEGUN, 1961]

$$\Psi(x) \equiv \frac{1}{\Gamma(x)} \cdot \frac{d\Gamma(x)}{dx} \tag{3.1-15}$$

to obtain the approximation $\Gamma(x+\delta) \approx \Gamma(x) \cdot [1+\delta \cdot \Psi(x)]$, with the result

$$\begin{aligned} r_{00} &= 2 \cdot \lim_{\delta \to 0^+} \left[\frac{[1+\delta \cdot \Psi(\mu_0+2)] \cdot [1+\delta \cdot \Psi(1)]}{[1+2 \cdot \delta \cdot \Psi(1)]} \right]^{1/2\delta} = \\ &= 2 \cdot \exp[\frac{\Psi(\mu_0+2)-\Psi(1)}{2}] \quad ; \end{aligned} \tag{3.1-16}$$

in particular, with $\mu_0 = 0$, we obtain the Critical Initial Displacement value

$$r_{0C} \equiv \lim_{\delta \to 0^+} P_E[\delta, 0] = 2 \cdot \exp[½] = 2\sqrt{e} = 3.2974... \quad . \tag{3.1-17}$$

Fig. 4.2-2 shows values of δ_0 and μ_0 as a function of the Initial displacement values r_{00}.

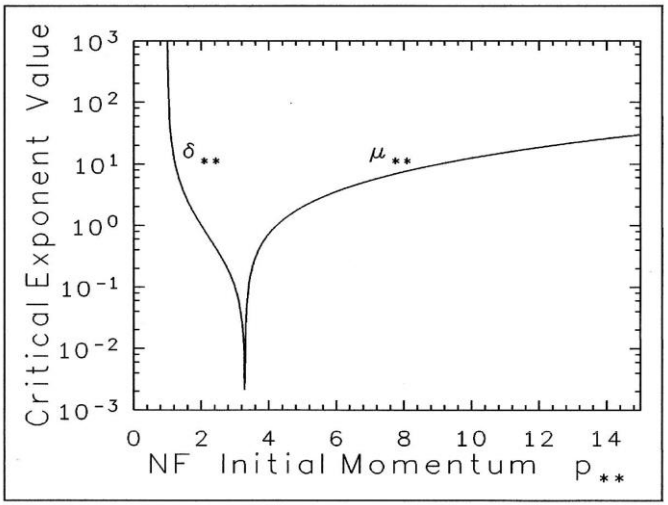


FIGURE 4.2-2 – CRITICAL VALUE OF DISPLACEMENT AND MOMENTUM EXPONENTS δ_{MIN} AND μ_{MIN} BELONGING WITH NF WD MAXIMUM DISPLACEMENT r_{00}

Table 4.2-1 shows the values of δ_{**} and μ_{**} for representative values of the NF WD Maximum Displacement r_{00}.

Table 4.2-1

r_{00}	δ_0	r_{00}	μ_0
1.00	∞	**5.75**	**3.13**
1.25	**8.90**	**5.50**	**2.74**
1.50	**3.28**	**5.25**	**2.36**
1.75	**1.70**	**5.00**	**2.00**
2.00	**1.00**	**4.75**	**1.65**
2.25	**0.615**	**4.50**	**1.33**
2.50	**0.375**	**4.25**	**1.02**
2.75	**0.214**	**4.00**	**0.728**
3.00	**0.0991**	**3.75**	**0.454**
3.25	**0.0138**	**3.50**	**0.196**
3.2974…	**0.00**	**3.2974…**	**0.00**

Figure 4.2-1 also shows the Dissipation Exponents δ and μ affect the Maximum Displacement r_{00} in different ways: For each particular Momentum Exponent value μ, the Maximum Displacement r_0 continually decreases in value as the Displacement Exponent value δ increases; conversely, for each particular Displacement Exponent value δ, the Maximum Displacement r_0 continually increases in value as the Momentum Exponent value μ increases

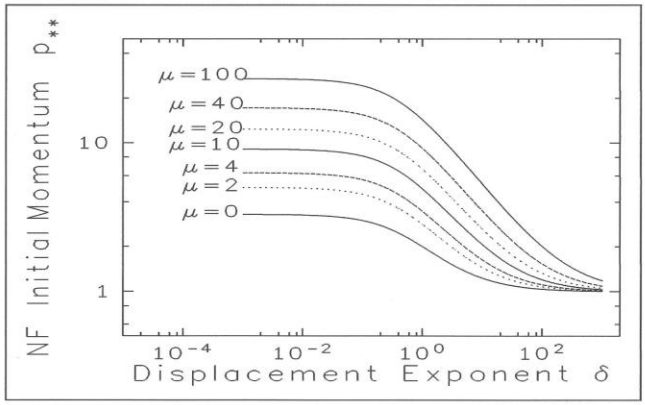


FIGURE 4.2-3 – INITIAL MOMENTUM p_{**} AS A FUNCTION OF DISPLACEMENT EXPONENT δ FOR PARTICULAR VALUES OF MOMENTUM EXPONENT μ

Figure 4.2-3 shows in more detail cross-sections of the NF Maximum Displacement r_0 shown in Fig. 4.2-1 as functions of the Displacement exponent δ for representative values of the Momentum exponent μ; it shows, in particular, $\lim_{\delta\to\infty} r_0 = 1.0$ for all values of the Momentum exponent μ, an asymptotic result we verify by expressing eq. (3.1-12) in the logarithmic form

$$Ln\left[r_0\right] = \frac{1}{2\delta}\left[Ln[\Gamma(\delta+\mu+2)] + Ln[\Gamma(\delta+1)]\right] + -\frac{1}{2\delta}\left[Ln[\Gamma(\mu+2)] - Ln[\Gamma(2\mu+1)]\right] \tag{3.1-18}$$

and using the leading term from the asymptotic representation [ABRAMOWITZ & STEGUN, 1961] $Ln[\Gamma(z)] \cong z \cdot Ln[z] - z - ½ \cdot Ln[z] + \ldots$ to obtain

$$\lim_{\delta\to\infty} Ln\left[r_0\right] = Ln[2] + \lim_{\delta\to\infty}\left[\begin{array}{l}\frac{\delta+\mu+2}{2\delta}Ln[\delta+\mu+2] + \\ +\frac{\delta+1}{2\delta}Ln[\delta+1]\end{array}\right] - \\ -\lim_{\delta\to\infty}\left[\begin{array}{l}\frac{\mu+2}{2\delta}Ln[\mu+2] + \\ +\frac{2\delta+1}{2\delta}Ln[(2\delta+1)]\end{array}\right] = (\mu+2)\lim_{\delta\to\infty}\left[\frac{Ln[\delta]}{2\delta}\right] = 0 \tag{3.1-19}$$

yielding the result $r_0[\infty,\mu] = 1.0$ for all Momentum exponent values μ.

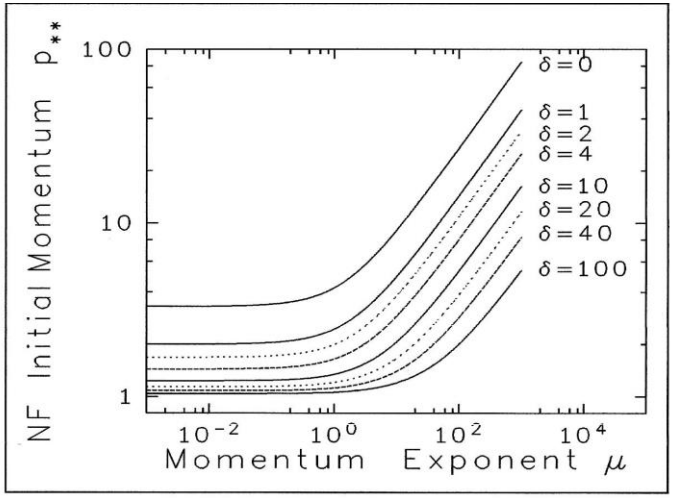


FIGURE 4.2-4 – INITIAL MOMENTUM p_{**} AS A FUNCTION OF MOMENTUM EXPONENT μ FOR PARTICULAR VALUES OF DISPLACEMENT EXPONENT δ

Figure 4.2-4 shows cross-sections of Fig. 4.2-1 for representative Displacement Exponent values δ; it shows, in particular, $p_{**} \propto \sqrt{\mu}$ for every Displacement exponent value δ, an asymptotic result we verify by using the leading term of the asymptotic result [TRICOMI & ERDELY; 1951]

$$\frac{\Gamma(x+a)}{\Gamma(x+b)} \approx x^{a-b}[1+\frac{(a-b)\cdot(a+b-1)}{2x}+...] , \tag{3.1-20}$$

to recover

$$p_{**} \approx 2\cdot\left[\mu^{\delta}\cdot\frac{\Gamma(\delta+1)}{\Gamma(2\delta+1)}\right]^{1/2\delta} \approx \sqrt{\mu}\cdot Q(\delta) \tag{3.1-21}$$

for all finite values of the Displacement exponent δ.

In the special case where the Momentum Exponent has an integer value $\mu = m > 0$, eq. (3.1-18) assumes the simpler form

$$\lim_{\delta\to 0} p_{**} = 2\cdot\exp\left[\frac{\mathrm{H}_{m+1}}{2}+[1+\frac{1}{3}+...+\frac{1}{2m-1}]\right] \tag{3.1-22}$$

with $H_{m+1} \equiv \sum_{k=1}^{m+1} k^{-1}$ as the $m+1$-st Harmonic Number. We also observe that the particular relationship $\delta = 1+\mu$ between Dissipation Function exponent values leads to the *remarkable result*

$$p_{**} = 2 \cdot \left[\frac{\Gamma(2\mu+3)}{\Gamma(\mu+2)} \cdot \frac{\Gamma(\mu+2)}{\Gamma(2\mu+3)} \right]^{1/2\delta} = 2 \tag{3.1-23}$$

for all values of the Momentum exponent μ. We then obtain a related asymptotic result in the limit $\mu \to \infty$ by using the representation $\delta = \rho \cdot \mu$ and re-writing eq. (3.1-18) as

$$\begin{aligned} Ln\left[p_{**}\right] &= Ln\left[2\right] + \\ &+ \frac{1}{2\rho\mu} \cdot \left[\begin{array}{l} Ln\left[\Gamma(\rho\mu+\mu+1)\right] + Ln\left[\Gamma(\rho\mu+1)\right] - \\ \quad - Ln\left[\Gamma(\mu+1)\right] - Ln\left[\Gamma(2\rho\mu+1)\right] \end{array} \right] . \end{aligned} \tag{3.1-24}$$

Upon using only the leading term in the expansion of $Ln[\Gamma(z)]$, we obtain

$$\lim_{\mu\to\infty} P_E[\rho\mu, \mu] \to \frac{[\rho+1]^{(\rho+1)/2\rho}}{\rho^{1/2}} , \tag{3.1-25}$$

together with the easily derived approximations

$$\begin{aligned} P_G[\rho] &\approx 1.5/\sqrt{\rho} \quad : \rho \le 2.25 , \\ P_G[\rho] &\approx 1.0 \qquad\quad : \rho \ge 2.25 . \end{aligned} \tag{3.1-26}$$

3.3 EVDP STRONG DISSIPATION BEHAVIOR

Under Strong Dissipation conditions ($\varepsilon \to \infty$), we elaborate the asymptotic LvdP analysis of Sec. 3.3 by representing the Warping Function as $W_\varepsilon(s) = \varepsilon^\Gamma B_\varepsilon(s)$, thereby allowing us to express the first two lines of eq. (3.1-5) as

$$\begin{aligned} \varepsilon^\Gamma B'_\varepsilon(s) &= -2 \cdot p_{**} Sin(s) \cdot \left[[p_{**}^2 Cos^2(s)]^\delta - 1 \right] \cdot \\ &\cdot \varepsilon^{(\Gamma+1)\cdot(\mu+½)} \left[\frac{p_{**}^2 Sin^2(s)}{\varepsilon^{\Gamma+1}} + B_\varepsilon(s) \right]^{\mu+½} ; \end{aligned} \tag{3.1-27}$$

upon then setting $\Gamma = (1+2\mu)/(1-2\mu)$, we can express eq. (3.1-27) in the simplified form

$$\begin{aligned} B'_\varepsilon(s) &= -2 \cdot p_{**} Sin(s) \cdot \left[[p_{**}^2 Cos^2(s)]^\delta - 1 \right] \cdot \\ &\cdot \left[\frac{p_{**}^2 Sin^2(s)}{\varepsilon^{\Gamma+1}} + B_\varepsilon(s) \right]^{\frac{\Gamma}{\Gamma+1}} . \end{aligned} \tag{3.1-28}$$

Thus, when $\mu \in [0, ½)$, so that $\Gamma + 1 > 0$, we then have

$$\begin{aligned} B'_\infty(s) &\equiv \lim_{\varepsilon \to \infty} B'_\varepsilon(s) = -2 \cdot p_{**} Sin(s) \cdot \left[[p_{**}^2 Cos^2(s)]^\delta - 1 \right] \cdot \\ &\cdot \lim_{\varepsilon \to \infty} \left[\frac{p_{**}^2 Sin^2(s)}{\varepsilon^{\Gamma+1}} + B_\varepsilon(s) \right]^{\frac{\Gamma}{\Gamma+1}} = \\ &= -2 \cdot p_{**} Sin(s) \cdot \left[[p_{**}^2 Cos^2(s)]^\delta - 1 \right] \cdot B_\infty^{\Gamma/\Gamma+1}(s) \end{aligned} \tag{3.1-29}$$

as the asymptotic representation of eq. (3.1-27); conversely, when $\mu \in (½, \infty)$, so that $\Gamma + 1 < 0$, the WTM cannot recover an asymptotic Warping Function representation. For this reason:

We presume $\mu \in (½, \infty)$ in the remainder of this Section.

Within the Initial Region $s \in [0, s_1)$, where $B'_\varepsilon(s) \le 0$, eq. (3.1-28) imposes a constraint

(3.1-30) $$0 \ge B_\infty(s) \ge -\lim_{\varepsilon \to \infty} \frac{p_{**}^2 Sin^2(s)}{\varepsilon^2} = 0 ,$$

for all values of Γ (i.e. all permitted values of the Momentum Exponent μ), whence we have $B_\infty(s_1) = 0$; this result also shows the asymptotic Intermediate Region to be empty because

(3.1-31) $$s_0 = s_1 = Cos^{-1}(p_{**}^{-1}) \ .$$

Within the Terminal Region, where $B_\varepsilon(s) \ge 0$, eq. (3.1-29) has the formal solution

(3.1-32) $$B_\infty^{1/(\Gamma+1)}(s) = \frac{2}{1+\Gamma}\left[p_{**}^{2\delta+1} \frac{[Cos(s)^2]^\delta \cdot Cos(s) + 1}{2\delta + 1} - p_{**}[Cos(s) + 1] \right]$$

automatically satisfying the required (terminal) boundary condition $B_\infty(\pi) = 0$ for all values of Γ Γ (i.e. all permitted values of the Momentum Exponent μ); the corresponding boundary condition $B_\infty(s_0) = 0 = B_\infty(s_1)$ then requires an Initial Momentum p_{**} satisfying the transcendental equation

(3.1-33) $$p_{**}^{2\delta+1} \frac{[Cos(s_0)^2]^\delta \cdot Cos(s_0) + 1}{2\delta + 1} - p_{**}[Cos(s_0) + 1] = 0 \ .$$

Upon using the definition of eq. (3.1-31), we can express eq. (3.1-33) as

(3.1-34) $$\frac{p_{**}^{2\delta} - 1}{2\delta} = \frac{p_{**} + 1}{p_{**}} ,$$

with the limiting form

(3.1-35) $$\lim_{\delta \to 0^+} \frac{p_{**}^{2\delta} - 1}{2\delta} = Ln[p_{**}] = \frac{p_{**} + 1}{p_{**}} ,$$

from which we obtain

$$\lim_{\delta \to 0^+} p_{**} = 3.5911... \quad . \tag{3.1-36}$$

At the other extreme, where $\delta \to \infty$, we obtain a useful bound on p_{**} by recalling that $p_{**} \geq 1$. We then express eq. (3.1-34) as

$$p_{**}^{2\delta+1} = 2\delta(p_{**}+1) + p_{**} \leq (2\delta+1)\cdot(p_{**}+1) \ , \tag{3.1-37}$$

from which it follows that

$$\begin{aligned} &\lim_{\delta \to \infty} Ln[p_{**}] \leq \lim_{\delta \to \infty} \frac{Ln[2\delta+1]}{2\delta+1} + \lim_{\delta \to \infty} \frac{Ln[p_{**}+1]}{2\delta+1} = 0 \ , \\ &\therefore \lim_{\delta \to \infty} p_{**} \to 1^+ \quad . \end{aligned} \tag{3.1-38}$$

Figure 4.3-1 shows how Initial Momentum p_{**} varies as a function of the Displacement exponent δ while Table 4.3-1 gives explicit values of Initial Momentum p_{**} for representative values of the Displacement exponent δ.

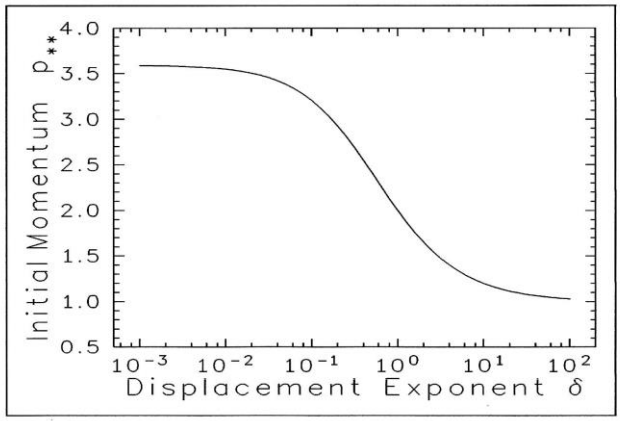


FIGURE 4.3-1 – ASYMPTOTIC ($\varepsilon \to \infty$) EVDP INITIAL MOMENTUM p_{**} AS A FUNCTION OF DISPLACEMENT EXPONENT VALUE δ FOR ALL MOMENTUM EXPONENT VALUES $\mu \in [0, ½)$

TABLE 4.3-1 – INITIAL MOMENTUM p_{**} FOR REPRESENTATIVE VALUES OF THE DISPLACEMENT EXPONENT δ

δ	p_{**}	δ	p_{**}	δ	p_{**}
0.0	**3.5911**	**0.1**	**3.2060**	**20.0**	**1.1146**
0.001	**3.5865**	**0.2**	**2.9271**	**50.0**	**1.0542**
0.002	**3.6820**	**0.5**	**2.4142**	**100.0**	**1.0303**
0.005	**3.5684**	**1.0**	**2.0000**	**200.0**	**1.0168**
0.01	**3.5491**	**2.0**	**1.6507**	**500.0**	**1.0076**
0.02	**3.5027**	**5.0**	**1.4918**	**1000.0**	**1.0041**
0.05	**3.3817**	**10.0**	**1.1989**	∞	**1.0000**

Table 4.2-1

p_{**}	$\delta_{**}[\varepsilon \to \infty]$	p_{**}	$\delta_{**}[\varepsilon \to 0^+]$
1.00	∞	**1.00**	∞
1.25	**7.45**	**1.25**	**8.90**
1.50	**2.93**	**1.50**	**3.28**
1.75	**1.61**	**1.75**	**1.70**
2.00	**1.00**	**2.00**	**1.00**
2.25	**0.654**	**2.25**	**0.615**
2.50	**0.434**	**2.50**	**0.375**
2.75	**0.282**	**2.75**	**0.214**
3.00	**0.171**	**3.00**	**0.0991**
3.25	**0.0867**	**3.25**	**0.0138**
3.591…	**0.00**	**3.2974…**	**0.00**

Table 4.2-1

p_{**}	δ_{**}	p_{**}	μ_{**}
1.00	∞	**5.75**	**3.13**
1.25	**8.90**	**5.50**	**2.74**
1.50	**3.28**	**5.25**	**2.36**
1.75	**1.70**	**5.00**	**2.00**
2.00	**1.00**	**4.75**	**1.65**
2.25	**0.615**	**4.50**	**1.33**
2.50	**0.375**	**4.25**	**1.02**
2.75	**0.214**	**4.00**	**0.728**
3.00	**0.0991**	**3.75**	**0.454**
3.25	**0.0138**	**3.50**	**0.196**
3.2974…	**0.00**	**3.2974…**	**0.00**

We recover the EvdP solution's period by using the last line of eq. (3.1-5) to define

(3.1-39) $$\Psi_{\varepsilon}(\pi) = 2 \cdot \int_0^{\pi} dx \frac{\left| p_{**} Sin(x; p_{**}) \right|}{\sqrt{p_{**}^2 Sin^2(x) + \varepsilon \cdot W_{\varepsilon}(x)}} \quad .$$

The Terminal Region, within which $\lim_{\varepsilon\to\infty}\varepsilon W_\varepsilon(s)=\varepsilon^{\Gamma+1}B_\infty(s)\to\infty$, and the (asymptotically empty) Transition Region make no contribution to the asymptotic period; conversely, upon using the representation

$$\varepsilon W_\varepsilon(s)=-p_{**}^2 Sin^2(s)+[\frac{e(s)}{\varepsilon^\Pi}]^2 \tag{3.1-40}$$

and the first line of eq. (3.1-5), we obtain

$$\begin{aligned}\varepsilon W'_\varepsilon(s)&=-2p_{**}^2 Sin(s)Cos(s)+2\frac{e(s)e'(s)}{\varepsilon^{2\Pi}}=\\&=-2\varepsilon p_{**}Sin(s)\Big[[p_{**}^2Cos^2(s)]^\delta-1\Big]\cdot[\frac{e(s)}{\varepsilon^\Pi}]^{1+2\mu}\ .\end{aligned} \tag{3.1-41}$$

as the Initial Region's contribution to eq. (3.1-39).

Upon defining $\Pi=1/(1+2\mu)$ and letting $\varepsilon\to\infty$, we recover

$$\begin{aligned}e^{1+2\mu}(s)&=\frac{p_{**}Cos(s)}{[p_{**}^2Cos^2(s)]^\delta-1}\ ,\\ \nu_\varepsilon(s)&=\left[\frac{1}{\varepsilon}\cdot\frac{p_{**}Cos(s)}{[p_{**}^2Cos^2(s)]^\delta-1}\right]^\Pi\end{aligned} \tag{3.1-42}$$

from which we obtain the *normalized* asymptotic period as

$$\lim_{\varepsilon\to\infty}\frac{\Psi_\varepsilon(\pi)}{\varepsilon^\Pi}=2\cdot\int_0^{Cos^{-1}(1/p_{**})}dx\ p_{**}Sin(x)\cdot\left[\frac{[p_{**}^2Cos^2(x)]^\delta-1}{p_{**}Cos(x)}\right]^\Pi\ . \tag{3.1-43}$$

For the particular case $\mu=0$, we obtain the closed form result

$$\frac{\Psi_\varepsilon(\pi)}{\varepsilon}=\frac{p_{**}^{2\delta}-1}{\delta}-2Ln[p_{**}]\ . \tag{3.1-44}$$

We can also use eq. (3.1-34) to express eq. (3.1-44) in the simpler form

$$\frac{\Psi_\varepsilon(\pi)}{\varepsilon}=2\left[\frac{p_{**}+1}{p_{**}}-Ln[p_{**}]\right], \tag{3.1-45}$$

with eq. (3.1-35) showing

$$\lim_{\delta \to 0^+} \frac{\Psi_\varepsilon(\pi)}{\varepsilon} = 2\left[Ln[p_{**}] - Ln[p_{**}] \right] = 0 \,, \tag{3.1-46}$$

and the last line of eq. (3.1-38) showing

$$\lim_{\delta \to \infty} \frac{\Psi_\varepsilon(\pi)}{\varepsilon} = 2\left[\frac{p_* + 1}{p_*} - Ln[p_{**}] \right] = 4 \ . \tag{3.1-47}$$

Figure 4.3-2 shows how the asymptotic period $\varepsilon^{-1}\mathrm{T}[\varepsilon] \equiv \Psi_\varepsilon(\pi)/\varepsilon$ defined by eq. (3.1-45) varies as a function of the Displacement exponent δ while Table 4.3-2 gives explicit values of the asymptotic period $\varepsilon^{-1}\mathrm{T}[\varepsilon]$ for representative values of the Displacement exponent δ.

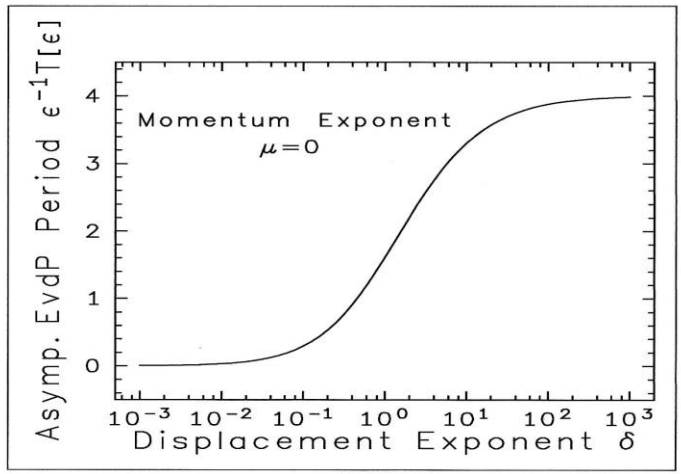


FIGURE 4.3-2 – ASYMPTOTIC ($\varepsilon \to \infty$) EVDP PERIOD AS A FUNCTION OF DISPLACEMENT EXPONENT VALUE δ FOR A MOMENTUM EXPONENT VALUE $\mu = 0$

TABLE 4.3-2 – ASYMPTOTIC ($\varepsilon \to \infty$) EVDP PERIOD FOR REPRESENTATIVE VALUES OF THE DISPLACEMENT EXPONENT δ

δ	$\varepsilon^{-1}\mathrm{T}[\varepsilon]$	δ	$\varepsilon^{-1}\mathrm{T}[\varepsilon]$	δ	$\varepsilon^{-1}\mathrm{T}[\varepsilon]$
0.0	**0.00000**	**0.1**	**0.29381**	**20.0**	**3.57729**
0.001	**0.00326**	**0.2**	**0.53525**	**50.0**	**3.79161**
0.002	**0.00653**	**0.5**	**1.06568**	**100.0**	**3.88147**
0.005	**0.01623**	**1.0**	**1.61366**	**200.0**	**3.93345**
0.01	**0.03232**	**2.0**	**2.20928**	**500.0**	**3.96978**
0.02	**0.06393**	**5.0**	**2.91095**	**1000.0**	**3.98357**
0.05	**0.15463**	**10.0**	**3.30525**	∞	**4.00000**

3.4 EVDP INTERMEDIATE DISSIPATION BEHAVIOR

For intermediate nonlinearity parameter values $\varepsilon \in (1,100)$, we use a numerical integration program [see Annex B] to examine the behavior of EvdP solutions exhibit periodic motion, with the most significant effect being -- as suggested by the asymptotic analysis of Sec. 4.3 -- a progressive curtailing of permitted Momentum exponent values μ as nonlinearity parameter value ε increases. Figure 4.4-1 illustrates this curtailment by using circles to superimpose the permitted Momentum exponent values μ associated with representative Initial Momentum values p_{**} given a nonlinearity parameter value $\varepsilon = 5$ upon the solid lines showing the corresponding (unconstrained) Weak Dissipation Momentum exponent values associated with $\varepsilon \to 0^+$, as given in Fig. 4.2-1 (we note, in particular, the absence of Momentum exponent values for $p_{**} = 5$ with a nonlinearity parameter value $\varepsilon = 5$).

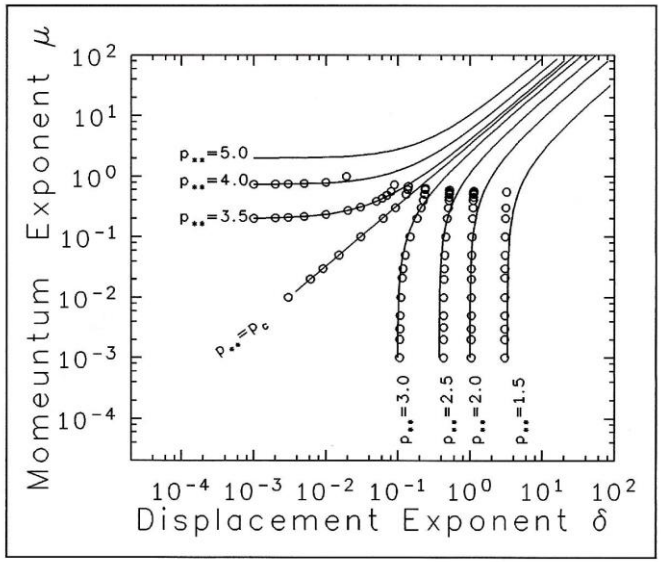


Figure 4.4-1 - Combinations of Dissipation Exponent values δ and μ yielding representative Initial Momentum values p_{**} [Solid Lines show $\varepsilon \to 0^+$; circles show $\varepsilon = 5$]

The progressive curtailment of Momentum exponent values as the nonlinearity parameter values increase, shown in Fig. 4.4-1, is due to the different ways in which the nonlinearity parameter value ε affects the Dissipation exponents, as illustrated by comparing their effects upon the Limit Cycles associated with a nonlinearity parameter value $\varepsilon = 1$

Chapter 4: WTM Analysis for the CvdP

Equation Chapter (Next) Section 1

4.1 Introduction to WTM for the CvdP

4.2 CvdP Weak Dissipation Behavior

4.3 CvdP Strong Dissipation Behavior

4.4 CvdP Intermediate Dissipation Behavior

Chapter 5: WTM Analysis for the vdPP

Equation Chapter (Next) Section 1

5.1 Introduction to WTM for the vdPP

5.2 vdPP Weak Dissipation Behavior

5.3 vdPP Strong Dissipation Behavior

5.4 vdPP Intermediate Dissipation Behavior

Annex A –LvdP Analyses

A.1 INITIAL LVDP ANALYSES

EQUATION SECTION (NEXT)

VAN DER POL'S ANALYSIS

VAN DER POL [1926] used the explicit absence of elapsed-time t in the LvdP

$$\ddot{u}(t)+\varepsilon\left[u^2(t)-1\right]\dot{u}(t)+u(t)=0 \qquad \text{A.1-1)}$$

to express momentum as a function $p[u]$ of displacement satisfying

$$p[u]\cdot\frac{dp[u]}{du}=\varepsilon\cdot(1-u^2)\cdot p[u]-u\ , \qquad \text{A.1-2)}$$

suitable for graphical solution via the Method of Isoclines [HAYASHI, 1964] while further observing that, for each of these nonlinearity parameter values, the asymptotic (u,p) phase-space trajectories approach a closed *Limit-Cycle* orbit, thereby permitting recovery of the periodic LvdP solutions shown in Fig. 1.1-1.

Van der Pol also provided the initial analytic LvdP characterization in the limit $\varepsilon\rightarrow 0^+$ by expressing the LvdP displacement as

$$u_0(t)=a(t)\cdot Cos[t+\phi] \qquad \text{A.1-3)}$$

with a slowly varying amplitude $\dot{a}(t)\ll a(t)$, Upon substituting the representation of eq. A.1-3) into eq. A.1-1) -- while "*omitting small quantities and neglecting higher harmonics*" -- van der Pol's analysis obtains

$$\frac{1}{\varepsilon}\dot{a}^2(t)-a^2(t)+\frac{1}{4}a^4(t)=0 \qquad \text{A.1-4)}$$

with the exact solution

A.1-5) $$a(t) = \frac{2 \cdot a_0}{\sqrt{a_0^2 + (4 - a_0^2) \cdot e^{-\varepsilon t}}}$$

for any initial displacement $a(0) \square a_0$; equation A.1-5) also shows the periodic LvdP solution to have a steady-state amplitude $\lim_{t \to \infty} a(t) = 2.0$ for all initial amplitude values a_0.

At the opposite extreme, when in the limit $\varepsilon \to \infty$ VAN DER POL [1927] obtains an approximate LvdP displacement period by observing that the periodic LvdP displacement $U_{10.0}(t)$ shown in Fig. 1.1-1 could be closely approximated between the initial value $U_{10.0}(t_I) = 2$ and the terminal value $U_{10.0}(t_T) = 1$ (corresponding to approximately one-half the full Period $T_0[10.0]$) by the solution[4]

A.1-6) $$Ln[u^2(t)] - u^2(t) = \frac{2t + C}{\varepsilon}$$

of eq. A.1-2)in the absence of its leading term. The result of eq. A.1-6) then expresses the full asymptotic LvdP Period for $\varepsilon \square 1$ as

A.1-7) $$\begin{aligned} T_0[\varepsilon] &= 2 \cdot \varepsilon \cdot [t_T - t_I] = \varepsilon \cdot \{[Ln(1) - 1] - [Ln(4) - 4]\} = \\ &= \varepsilon \cdot \{3 - 2 \cdot Ln(2)\} = 1.613... \cdot \varepsilon \ . \end{aligned}$$

KRYLOV & BOGOLIUBOF'S ANALYSIS

KRYLOV & BOGOLIUBOF [1937] provide a more elaborate LvdP analysis for nonlinearity parameter values in the limit $\varepsilon \to 0^+$ by expressing the LvdP displacement in a form

A.1-8) $$u_0(t) = a(t) \cdot Cos[t + \phi(t)]$$

permitting both the amplitude and phase to be slowly time-varying functions of elapsed-time t; their analysis also *mandates* a linear linkage

A.1-9) $$\dot{a}(t) \cdot Cos[t + \phi(t)] - a(t) \cdot \dot{\phi}(t) \cdot Sin[t + \phi(t)] = 0$$

---

[4] This analysis suggests that van der Pol *presumed* (possibly due to the limits of graphical integration used in constructing the results shown in Fig. 1.1-1) a maximum displacement value $U_0[\varepsilon] \equiv 2$ for all values of the nonlinearity parameter ε.

between the slowly-varying rates $\dot{a}(t)$ and $a(t)\cdot\dot{\phi}(t)$, so that

$$p_0(t) = -a(t)\cdot Sin[t+\phi(t)] \quad . \qquad \text{A.1-10)}$$

After differentiating the LvdP momentum representation of eq. A.1-10) with respect to elapsed-time t to obtain the full LvdP acceleration

$$\ddot{u}_0(t) = -\dot{a}(t)Sin[t+\phi(t)] - a(t)Cos[t+\phi(t)]\cdot[1+\dot{\phi}(t)] \,, \qquad \text{A.1-11)}$$

Krylov & Bogoliubof continue by substituting the representations of LvdP displacement, momentum and acceleration given by eqs. A.1-8), A.1-10), and A.1-11) into the LvdP eq. **Error! Reference source not found.**, thereby obtaining a second linear linkage

$$\begin{aligned} &\dot{a}(t)\cdot Sin[t+\phi(t)] + a(t)\cdot\dot{\phi}(t)\cdot Cos[t+\phi(t)] = \\ &= -\varepsilon\left[a^2(t)\cdot Cos^2[t+\phi(t)]-1\right]\cdot a(t)\cdot Sin[t+\phi(t)] \end{aligned} \qquad \text{A.1-12)}$$

between the slowly-varying rates $\dot{a}(t)$ and $a(t)\cdot\dot{\phi}(t)$.

Equations A.1-9) and A.1-12) now provide two independent expressions relating the slowly time-varying quantities $\dot{a}(t)$ and $a(t)\cdot\dot{\phi}(t)$, from which it follows that

$$\begin{aligned} \dot{a}(t) &= -\varepsilon\left[a^2(t)\cdot Cos^2[t+\phi(t)]-1\right]\cdot a(t)\cdot Sin^2[t+\phi(t)] \,, \\ \dot{\phi}(t) &= \varepsilon\left[a^2(t)\cdot Cos^2[t+\phi(t)]-1\right]\cdot Sin[t+\phi(t)]\cdot Cos[t+\phi(t)] \quad . \end{aligned} \qquad \text{A.1-13)}$$

The slowly-varying behavior $\dot{a}(t)$ and $a(t)\cdot\dot{\phi}(t)$ then allows Krylov & Bogoliubof to replace the right-hand sides of eq. A.1-13) with values averaged over one full cycle of elapsed-time, i.e.

$$\begin{aligned} \dot{a}(t) &\approx -\varepsilon\int_0^{2\pi} dt\left[a^2(t)\cdot Cos^2[t+\phi(t)]-1\right]\cdot \\ &\qquad \cdot a(t)\cdot Sin^2[t+\phi(t)] = \varepsilon\cdot\frac{a(t)}{2}\cdot\left[1-\frac{a^2(t)}{4}\right] \,, \\ \dot{\phi}(t) &\approx \varepsilon\int_0^{2\pi} dt\left[a^2(t)\cdot Cos^2[t+\phi(t)]-1\right]\cdot \\ &\qquad \cdot Sin[t+\phi(t)]\cdot Cos[t+\phi(t)] = 0 \quad , \end{aligned} \qquad \text{A.1-14)}$$

The first line of eq. A.1-14) now replicates (with different notation) the amplitude variation previously given by van der Pol in eq. A.1-4), while the second line of eq. A.1-14) similarly validates van der Pol's use of a constant phase.

A.2 INTERMEDIATE LVDP ANALYSES

EQUATION SECTION (NEXT)

INTERMEDIATE LVDP ANALYSES

Intermediate LvdP analyses obtain results for moderate nonlinearity parameter values $\varepsilon \approx 1$ by expressing the displacement as a summation of terms,

$$u_\varepsilon(t) = u_N(t) + \sum_{n=1}^{\infty} \varepsilon^n \cdot u_n(t) \tag{A.2-1}$$

each multiplied by successively larger powers of the nonlinearity parameter ε. These intermediate LvdP analyses differ in the ways they express and recover the terms $u_n(t)$, but their representations of periodic, steady-state displacements are indistinguishable. In the following paragraphs, we summarize these intermediate LvdP analyses and illustrate their indistinguishable behavior by recovering their periodic, steady-state displacements of order ε^2.

Perturbation analyses seek steady-state solution of eq. **Error! Reference source not found.** with $\varepsilon \ll 1$ by adding a small, first-order *perturbation* $\varepsilon \cdot w_1(t)$ to the LvdP NF solution, i.e.

$$u_\varepsilon(t) = u_N(t) + \varepsilon \cdot w_1(t) \ , \tag{A.2-2}$$

with $w_1(t)$ sharing the periodicity of the NF solution $u_N(t)$. Upon substituting eq. (A.2-2) into the LvdP eq. **Error! Reference source not found.** and equating terms containing ε, we obtain the first-order perturbation equation

$$\ddot{w}_1(t) + w_1(t) = [1 - u_N^2(t)] \cdot \dot{u}_N(t) \quad . \tag{A.2-3}$$

For the representative NF solution $u_N(t) = A \cdot Cos(t)$ used by van der Pol, eq. (A.2-3) yields

$$\ddot{w}_1(t) + w_1(t) = [A^2 \cdot Cos^2(t) - 1] \cdot A \cdot Sin(t) = \\ = (\frac{A^2}{4} - 1) \cdot A \cdot Sin(t) + \frac{A^3}{4} \cdot Sin(3 \cdot t) \quad . \tag{A.2-4}$$

The adjustable parameter value $A = 2$, required to preclude the appearance of a secular term in $w_1(t)$, then yields the perturbed solution

$$u_\varepsilon(t) = 2 \cdot Cos(t) - \frac{\varepsilon}{4} \cdot Sin(3 \cdot t) \quad . \tag{A.2-5}$$

It is, however, not generally possible to continue this process, e.g. by adding a second-order perturbation $\varepsilon^2 \cdot w_2(t)$ term yielding

$$u_\varepsilon(t) = u_N(t) + \varepsilon \cdot w_1(t) + \varepsilon^2 \cdot w_2(t) \tag{A.2-6}$$

with $w_2(t)$ sharing the periodicity of $u_N(t)$ and $w_1(t)$, because there is no adjustable parameter in the second –order perturbation equation

$$\ddot{w}_2(t) + w_2(t) = -[u_N^2(t) - 1] \cdot \dot{w}_1(t) - 2 \cdot w_1(t) \cdot u_N(t) \cdot \dot{u}_N(t) = \\ = \frac{5}{4} \cdot Cos(5 \cdot t) + [\frac{1}{4}] \cdot Cos(t) - \frac{3}{2} \cdot Cos(3 \cdot t) \; ; \tag{A.2-7}$$

permitting us to preclude the appearance of a secular term in $w_2(t)$.

$$\ddot{w}_2(t) + w_2(t) = [1 - u_N^2(t) - 1] \cdot \dot{w}_1(t) - 2 \cdot w_1(t) \cdot u_N(t) \cdot \dot{u}_N(t) = \tag{A.2-8}$$

w

[SHOHAT, 1944] recover the periodic, by expressing elapsed-time

in terms of the nonlinearity parameter ε as a summation

(A.2-9) $$U_\varepsilon(\tau) = u_N(\tau) + \sum_{n=1}^{\infty} \varepsilon^n \cdot w_n(\tau) \ ,$$

of individual periodic terms $w_n(\tau)$, each expressed as a function of an auxiliary-time τ and satisfying

(A.2-10) $$\begin{aligned} &w_n(\tau + 2\pi) - w_n(\tau) = 0 \ , \\ &w_n'(0) = 0 \ ; \end{aligned}$$

he similarly defines elapsed-time

(A.2-11) $$t = c(\varepsilon) \cdot \tau = [1 + \sum_{n=1}^{\infty} c_n \cdot \varepsilon^n] \cdot \tau$$

as a 'scaled' version of the auxiliary-time τ and expresses eq. **Error! Reference source not found.** as

(A.2-12) $$u_\varepsilon''(\tau) + \varepsilon\left[u_\varepsilon^2(\tau) - 1\right] \cdot c(\varepsilon) \cdot u_\varepsilon'(\tau) + c^2(\varepsilon) \cdot u_\varepsilon(\tau) = 0 \ ,$$

while using accents denote differentiation with respect to the auxiliary-time τ.

This analysis continues by substituting the periodic solution $U_\varepsilon(\tau)$ defined by eq. (A.2-9) into eq. (A.2-12) and equating terms incorporating equal power of the nonlinearity parameter ε to obtain

(A.2-13) $$\begin{aligned} &\left[w_0''(\tau) + w_0(\tau)\right] \cdot \varepsilon^0 + \\ &+\left[w_1''(\tau) + w_1(\tau) + [u_0^2(\tau) - 1] \cdot u_0'(\tau) + 2 \cdot c_1 \cdot u_0(\tau)\right] \cdot \varepsilon + \\ &+\begin{bmatrix} w_2''(\tau) + w_2(\tau) + [u_0^2(\tau) - 1] \cdot w_1'(\tau) + \\ +2 \cdot w_1(\tau) \cdot u_0(\tau) \cdot u_0'(\tau) + 2c_2 \cdot u_0(\tau) \end{bmatrix} \varepsilon^2 + \ldots = 0 \ ; \\ &\qquad \square \quad \square \quad \square \end{aligned}$$

in particular, the n-th order perturbation equation (i.e. the term of eq. (A.2-13) multiplied by ε^n) has a form

(A.2-14) $$w_n''(\tau) + w_n(\tau) = F[w_0(\tau), w_1(\tau), \ldots, w_{n-1}(\tau)] \ ,$$

emulating that of the NF eq.**Error! Reference source not found.**, but now excited by the solutions of lower order. Thus, upon beginning with the complete zero-order solution

$$w_0(\tau) = a \cdot Cos(\tau) \tag{A.2-15}$$

satisfying the initial conditions of eq. (A.2-10), but with the value of a to be determined subsequently, Shohat's analysis expresses the first order perturbation equation in the explicit form

$$\begin{aligned} w_1''(\tau) + w_1(\tau) &= -[u_0^2(\tau) - 1] \cdot u_0'(\tau) - 2 \cdot c_1 \cdot u_0(\tau) = \\ &= -[\frac{a^3}{4} - a] \cdot Sin(\tau) + \frac{a^3}{4} \cdot Sin(3\tau) - 2 \cdot c_1 \cdot a \cdot Cos(\tau) \ . \end{aligned} \tag{A.2-16}$$

The particular values $a = 2$ and $c_1 = 0$ now ensure the absence of non-periodic *secular* terms in the first-order forced-solution

$$\hat{w}_1(\tau) = -\frac{1}{4} Sin(3\tau) \tag{A.2-17}$$

of eq. (A.2-16); addition of a particular homogeneous solution

$$\tilde{w}_1(\tau) = \frac{3}{4} \cdot Sin(\tau) + b_1 \cdot Cos(\tau) \tag{A.2-18}$$

then yields the complete first-order solution

$$w_1(\tau) = -\frac{1}{4} Sin(3\tau) + \frac{3}{4} \cdot Sin(\tau) + b_1 \cdot Cos(\tau) \tag{A.2-19}$$

(with the value of b_1 determined subsequently) satisfying the initial conditions of eq. (A.2-10).

In the same way, Shohat's analysis expresses the second-order equation in the explicit form

$$\begin{aligned} w_2''(s) + w_2(s) &= -[u_N^2(s) - 1] \cdot w_1'(s) - \\ &\quad - 2 \cdot w_1(s) \cdot u_N(s) \cdot u_N'(s) - 2c_2 \cdot u_N(s) = \\ &= \frac{5}{4} \cdot Cos(5s) + [\frac{1}{4} - 4c_2] \cdot Cos(s) - \frac{3}{2} \cdot Cos(3s) \ . \end{aligned} \tag{A.2-20}$$

The particular values $b_1 = 0$ and $c_2 = 1/16$ now ensure the absence of secular terms in the second-order forced-solution

(A.2-21) $$\hat{w}_2(s) = \frac{3}{16} \cdot Cos(3s) - \frac{5}{96} \cdot Cos(5s)$$

of eq. (A.2-20), while addition of the particular homogeneous solution

(A.2-22) $$\tilde{w}_2(s) = -\frac{1}{8} \cdot Cos(s)$$

(determined by a third-order analysis not reproduced here) yields the compete second-order solution

(A.2-23) $$w_2(s) = \frac{3}{16} \cdot Cos(3s) - \frac{5}{96} \cdot Cos(5s) \; - \frac{1}{8} \cdot Cos(s)$$

satisfying the initial conditions of eq. (A.2-10). Taken together, eqs. (A.2-15), (A.2-19), and (A.2-23) yield the complete second-order periodic LvdP solution

$$\begin{aligned} U_\varepsilon(t) &= 2 \cdot Cos(\omega \cdot t) + \frac{\varepsilon}{4} \cdot \left[3 \cdot Sin(\omega \cdot t) - Sin(3 \cdot \omega \cdot t)\right] + \\ &+ \frac{\varepsilon^2}{96}\left[-12 \cdot Cos(\omega \cdot t) + 18 \cdot Cos(3 \cdot \omega \cdot t) - 5 \cdot Cos(5 \cdot \omega \cdot t)\right] \;, \end{aligned} \tag{A.2-24}$$

with eq. (A.2-11) supplying the corresponding second-order radian frequency

$$\omega \square \frac{1}{c(\varepsilon)} = 1 - \frac{\varepsilon^2}{16} \;. \tag{A.2-25}$$

Proceeding in the same way, it is possible to recover the nth-order LvdP solution for any value of n, and BUONOMO [1998A] provides recurrent analytical formulae for this purpose; this expansion is, however, limited to values of the nonlinearity parameter ε not exceeding a critical limit $\varepsilon_C \approx 1.89$.

Figures 1.3-1 through 1.3-3 illustrate the LvdP displacement's evolution for a nonlinearity parameter value $\varepsilon = 1$ through the first three ap-

proximation steps, shown as dashed lines, with the solid line depicting the exact solution $U_{1.0}(t)$.

AN HISTORICAL NOTE

It is possible to obtain convergence of the LvdP series expansion shown in eq. **(A.2-1)** for values of the nonlinearity parameter ε larger than the critical limit $\varepsilon_C \approx 1.89$ .by using an expansion parameter expressed as a function $q[\varepsilon]$ of the nonlinearity parameter ε instead of the nonlinearity parameter itself; in particular, the representation $q[\varepsilon] \equiv \varepsilon/(1+\varepsilon)$ allows eq. (A.2-25) to be expressed as

$$[1+\varepsilon]\cdot\omega \equiv 1+q+\frac{15}{16}\cdot q^2+\frac{13}{16}\cdot q^3 \ , \qquad \therefore \ \lim_{\varepsilon\to\infty}\varepsilon\cdot\omega \to 3.75 \ , \tag{A.2-26}$$

while the asymptotic result of eq. A.1-7) gives

$$\lim_{\varepsilon\to\infty}\varepsilon\cdot\omega = \frac{2\pi}{3-2\cdot Ln(2)} = 3.89 \ . \tag{A.2-27}$$

Given the similarity of results in eqs. (A.2-26) and (A.2-27), Shohat (erroneously) conjectured that the representation of eq. (A.2-9), with $\varepsilon \equiv q[\varepsilon]/(1-q[\varepsilon])$ might converge for all values of the nonlinearity parameter ε; subsequent analysis [DE BRUIJN, 1945/46] however demonstrates this conjecture to be erroneous, with BUONOMO [1998A] recovering a critical value $\varepsilon_C \approx 2.70$. BUONOMO [1998B] shows, however, an expansion with $q[\varepsilon] \equiv \varepsilon^2/(1+\varepsilon^2)$ converges for all values of the nonlinearity parameter ε.

THE KBM METHOD

The KBM [Krylov-Bogoliubov-Mitropolsky] Method [MEIROVITCH; 1970] recasts the analysis of Krylov & Bogoliubof by following Shohat in using the summation of terms $w_n[a,\psi]$, each multiplied by increasing powers of the nonlinearity parameter ε, to represent the LvdP solution of eq. **Error! Reference source not found.** as a function amplitude a and phase $\psi \equiv t+\phi$, defined in terms of elapsed-time t by

$$\dot{a} = \sum_{n=1}^{\infty} \varepsilon^n A_n[a] \ ,$$
$$\dot{\psi} = 1 + \sum_{n=1}^{\infty} \varepsilon^n B_n[a] \ . \tag{A.2-28}$$

In particular, the KBM Method represents the LvdP solution in a form, analogous to that of eq. (A.2-9),

$$u_\varepsilon(t) \square u_\varepsilon[a;\psi] = u_{\mathrm{N}}[a,\psi] + \sum_{n=1}^{\infty} \varepsilon^n w_n[a;\psi] \tag{A.2-29}$$

with the functions $w_n[a;\psi]$ subject to the boundary conditions

$$w_n[a, s+2\pi] = w_n[a,s] \ ,$$
$$w_n'[a,0] = 0 \ . \tag{A.2-30}$$

Upon substituting the definition of $u_\varepsilon(t)$ given by eq. (A.2-29) into the LvdP eq. **Error! Reference source not found.**, equating terms containing ε, and using the chain rule

$$\frac{d}{dt} = \dot{\psi} \cdot \frac{\partial}{\partial \psi} + \dot{a} \cdot \frac{\partial}{\partial a} = \frac{\partial}{\partial \psi} + \varepsilon \cdot \left[B_1[a] \cdot \frac{\partial}{\partial \psi} + A_1[a] \cdot \frac{\partial}{\partial a} \right] , \tag{A.2-31}$$

the KBM Method recovers the first-order perturbation equation

$$\frac{\partial^2 w_1[a;\psi]}{\partial \psi^2} + w_1[a;\psi] = 2B_1[a] \cdot a \cdot Cos[\psi] +$$
$$+2A_1[a] \cdot Sin[\psi] + \left(\frac{a^2}{4} - 1 \right) \cdot a \cdot Sin[\psi] + \frac{a^3}{4} \cdot Sin[3\psi] \ ; \tag{A.2-32}$$

the definitions

$$A_1[a] = \frac{a \cdot [4 - a^2]}{8} \ , \ B_1[a] = 0 \ , \tag{A.2-33}$$

now ensure the forced solution

(A.2-34) $$\hat{w}_1[a,\psi] = -\frac{a^3}{32} \cdot Sin[3\psi]$$

of eq. (A.2-32) has the periodicity of $Cos[\psi]$, while simultaneously yielding the first-order results

(A.2-35) $$\dot{a} = \varepsilon A_1[a] = \varepsilon \cdot \frac{a}{2} \cdot [1 - \frac{a^2}{4}] \,, \qquad \dot{\psi} = 1 \,,$$

obtained previously in eq. A.1-14), but without requiring the mandated constraint of eq.A.1-9).

Proceeding in the same way, the KBM Method obtains the second-order LvdP solution by equating terms containing ε^2 and using the chain-rule representation

(A.2-36) $$\frac{d}{dt} = \frac{\partial}{\partial \psi} + \varepsilon \cdot A_1[a] \cdot \frac{\partial}{\partial a} + \varepsilon^2 \cdot \left[B_2[a] \cdot \frac{\partial}{\partial \psi} + A_2[a] \cdot \frac{\partial}{\partial a} \right] ,$$

to recover

$$\begin{aligned} &\frac{\partial^2 w_2[a,\psi]}{\partial \psi^2} + w_2[a,\psi] = 2 \cdot A_2[a] \cdot Sin[\psi] + \\ &+ \left[\begin{array}{l} 2 \cdot B_2[a] \cdot a - A_1[a] \cdot \dfrac{dA_1[a]}{da} + \\ +[1 - \dfrac{3 \cdot a^2}{4}] \cdot A_1[a] + \dfrac{a^5}{128} \end{array} \right] \cdot Cos[\psi] \\ &+ \left[2 \cdot A_1[a] + [\frac{3 \cdot a^5}{64} - \frac{5 \cdot a^3}{32}] \right] \cdot Cos[3\psi] + \left[\frac{5 \cdot a^5}{128} \right] \cdot Cos[5\psi] \;; \end{aligned} \tag{A.2-37}$$

the definitions

$$A_2[a] = 0 \,, \;\; B_2[a] = -\frac{8 - 41 \cdot a^2 + 10 \cdot a^4}{64} \,, \tag{A.2-38}$$

now ensure a forced solution of eq. (A.2-37) having the periodicity of $Cos[\psi]$; the third-order analysis (not included here) further requires ap-

pending a particular term $-B_2[a]\cdot Cos[\psi]/4$, satisfying the homogeneous version of eq. (A.2-37), to obtain the complete second-order solution

$$w_2[a,\psi]=\frac{8-41\cdot a^2+10\cdot a^4}{256}\cdot Cos[\psi]-$$
$$-a\cdot\frac{16-10\cdot a^2+3\cdot a^4}{512}\cdot Cos[3\psi]-\frac{5\cdot a^5}{3072}\cdot Cos[5\psi]\ . \qquad \text{(A.2-39)}$$

In the steady-state, where the definition $\dot{a}=\varepsilon\cdot A_1[a]=0$ requires $a=2$ and $B_2[2]=-1/16$, the results given in eqs. (A.2-34) and (A.2-39) lead to the complete *periodic* second-order LvdP solution in the explicit form

$$U_\varepsilon(t)=2\cdot Cos[\omega t]-\varepsilon\cdot\frac{1}{4}\cdot Sin[3\omega t]+$$
$$+\varepsilon^2\cdot\left[\frac{1}{64}\cdot Cos[\omega t]-\frac{3}{32}\cdot Cos[3\omega t]-\frac{5}{96}\cdot Cos[5\omega t]\right] \qquad \text{(A.2-40)}$$
$$\omega\cdot\dot{\psi}=\left[1-\frac{1}{16}\cdot\varepsilon^2\right]\ .$$

The periodic second-order LvdP solution given by eq. (A.2-40) differs, however, from that of the Shohat analysis shown in eq. (A.2-24) **Error! Reference source not found.**, by yielding the particular result $\dot{U}_\varepsilon(0)=-3\cdot\omega\cdot\varepsilon/4$ instead of $\dot{U}_\varepsilon(0)=0$; upon substituting

(A.2-41) $$\psi=\left[1-\frac{1}{16}\cdot\varepsilon^2\right]\cdot t-\frac{3}{8}\varepsilon\cdot\omega t-\frac{3}{8}\cdot\varepsilon$$

into each term of eq. (A.2-40), however, while retaining only second order terms and using appropriate trigonometric identities, the KBM Method solution expressed by eq. (A.2-40) becomes

(A.2-42) $$U_\varepsilon(t)=2\cdot Cos[\omega t]+\frac{\varepsilon}{4}\cdot\left[3\cdot Sin[\omega t]-Sin[3\cdot\omega t]\right]$$
$$+\frac{\varepsilon^2}{96}\cdot\left[-12\cdot Cos[\omega t]+18\cdot Cos[3\omega t]-5\cdot Cos[5\omega t]\right]$$

exactly matching the second-order Series solution given in eq. (A.2-24) **Error! Reference source not found.**.

The KBM Method also recovers the complete second-order transient LvdP solution with $a(0) = a_0$ and $\psi(0) = 0$ by using eq. (A.2-28) to recover the amplitude

$$a(t) = \frac{2 \cdot a_0}{\sqrt{a_0^2 + (4 - a_0^2) \cdot e^{-\varepsilon t}}} \ , \qquad \text{(A.2-43)}$$

given by van der Pol's analysis, shown in eq. A.1-5) **Error! Reference source not found.**; it also recovers the time-varying frequency (not previously recovered)

$$\begin{aligned} \omega \square \ \dot{\psi}[a] &= 1 - \varepsilon^2 \cdot B[a] = \\ &= 1 - \varepsilon^2 \cdot \frac{8 - 41 \cdot a^2(t) + 10 \cdot a^4(t)}{64} \ , \end{aligned} \qquad \text{(A.2-44)}$$

shown in Fig. 1.2-4 for the case $\varepsilon = 0.1$ and $a_0 = 1.0$. In the same way, Fig. 1.2-5 shows the complete second-order LvdP transient solution, analogous to that shown in Fig. 1.1-1.

BURTON'S TIME-TRANSFORM ANALYSIS (TTA)

BURTON [1981] expresses elapsed-time as a function

(A.2-45) $$t = \int_0^{\tau} \frac{dx}{f_\varepsilon(x)} \ ,$$

of an auxiliary-time τ and seeks, for each particular value of the nonlinearity parameter ε, the function $f_\varepsilon(x)$ and accompanying amplitude a_ε such that

(A.2-46) $$U_\varepsilon(t) = u_N(\tau) = a_\varepsilon \cdot Cos(\tau)$$

is a periodic solution of eq. **Error! Reference source not found.**. In doing this, the TTA first uses the representations of eqs. (A.2-45) and (A.2-46) to express the LvdP eq. **Error! Reference source not found.** as

$$\begin{aligned}&[f_\varepsilon^2(\tau)-1]\cdot Cos(\tau)+f_\varepsilon(\tau)\cdot f_\varepsilon'(\tau)\cdot Sin(\tau)-\\&-\varepsilon\cdot f_\varepsilon(\tau)\cdot[1-a_\varepsilon^2\cdot Cos^2(\tau)]\cdot Sin(\tau)=0\ ,\end{aligned} \tag{A.2-47}$$

after which, it substitutes the representations

$$\begin{aligned}a_\varepsilon^2&=4\sum_{n=0}^{\infty}\varepsilon^n\cdot\delta_n\ ,\\f_\varepsilon(\tau)&=\sum_{n=0}^{\infty}\varepsilon^n\cdot\gamma_n(\tau)\ ,\end{aligned} \tag{A.2-48}$$

into eq. (A.2-47) and equates terms containing common powers of the nonlinearity parameter ε. The TTA next satisfies the zero-order representation

$$[\gamma_0^2(\tau)-1]\cdot Cos(\tau)+\gamma_0(\tau)\cdot\gamma_0'(\tau)\cdot Sin(\tau)=0 \tag{A.2-49}$$

by defining $\gamma_0(\tau)=1$ and similarly satisfies the first-order representation

$$2\gamma_1(\tau)\cdot Cos(\tau)+\gamma_1'(\tau)\cdot Sin(\tau)=(1-\delta_0)\cdot Sin(\tau)-\delta_0\cdot Sin(3\cdot\tau) \tag{A.2-50}$$

by defining $\delta_0=1$ with

$$\gamma_1(\tau)=-\frac{Sin(2\cdot\tau)}{2}\ . \tag{A.2-51}$$

Continuing in the same way, the TTA eventually recover (through analyses not included here) the second-order representation

$$\gamma_2(\tau)=\frac{[3-2\cdot Cos(2\cdot\tau)-Cos(4\cdot\tau)]}{48} \tag{A.2-52}$$

and uses the results of eqs. (A.2-51) and (A.2-52) to recover

$$f_\varepsilon(x) = 1 - \varepsilon \cdot \frac{Sin(2 \cdot s)}{2} + \varepsilon^2 \cdot \frac{3 - 2 \cdot Cos(2 \cdot s) - Cos(4 \cdot s)}{48},$$
$$a^2 = 4 \cdot [1 + \varepsilon^2 \cdot \frac{1}{24}], \qquad \text{(A.2-53)}$$

with eq. (A.2-45) then providing the second-order representation

$$t[\tau] = \int_0^s \frac{dx}{f_\varepsilon(x)} = \left[1 + \frac{\varepsilon^2}{16}\right] \cdot \tau - \frac{\varepsilon}{4} \cdot \left[Cos(2 \cdot \tau) - 1\right] + \frac{\varepsilon^2}{48} \cdot \left[Sin(2 \cdot \tau) - \frac{5}{4} \cdot Sin(4 \cdot \tau)\right]. \qquad \text{(A.2-54)}$$

Figure 1.3-6 shows the evolution of $t[s]$ as a function of the representation order. Figures 1.3-7 through 1.3-9 show the corresponding periodic LvdP solutions $U_{1.0}(t)$ with the solid line showing the exact solution and +'s showing the LvdP solutions obtained through the TTA. It also follows from eq. (A.2-54) that the LvdP solution's second-order period and radian frequency

$$T_0[\varepsilon] = 2\pi \cdot [1 + \frac{\varepsilon^2}{16}],$$
$$\omega \square \frac{2\pi}{T_0[\varepsilon]} = 1 - \frac{\varepsilon^2}{16}, \qquad \text{(A.2-55)}$$

replicate the Fourier Series and KBM Method results given respectively by eqs.(A.2-25) and (A.2-40); in addition, upon defining $\psi = \omega \cdot t$ and inverting eq. (A.2-54)to recover

$$\tau = \psi + \frac{\varepsilon}{4} \cdot Cos[2 \cdot \psi] - \frac{\varepsilon^2}{48} \cdot \left[Sin[2 \cdot \psi] + \frac{7}{4} \cdot Sin[4 \cdot \psi]\right], \qquad \text{(A.2-56)}$$

the second-order periodic LvdP solution defined by eq. (A.2-46) can be expressed as

$$U_\varepsilon(t) = a_\varepsilon \cdot Cos(\tau) = 2 \cdot Cos[\omega t] + \frac{\varepsilon}{4} \cdot \left[3 \cdot Sin[\omega t] - Sin[3 \cdot \omega t]\right] + \frac{\varepsilon^2}{96} \cdot \left[-12 \cdot Cos[\omega t] + 18 \cdot Cos[3\omega t] - 5 \cdot Cos[5\omega t]\right] \qquad \text{(A.2-57)}$$

exactly matching the second-order LvdP solutions given in eqs. (A.2-24) and (A.2-42).

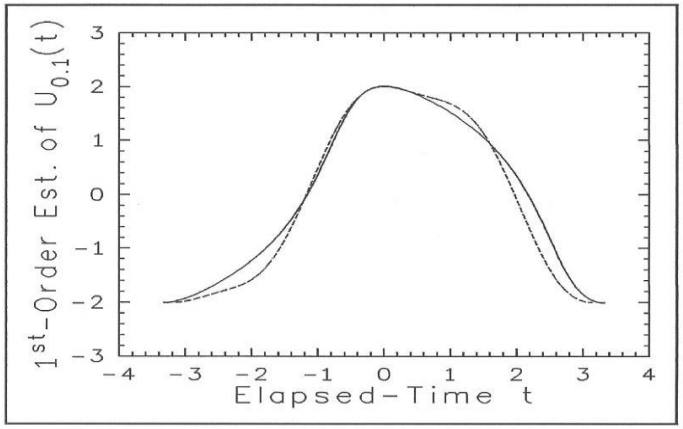


FIGURE 1.3-1 – 0TH-ORDER ESTIMATE

$$U_{1.0}(t) = w_0(\tau)$$

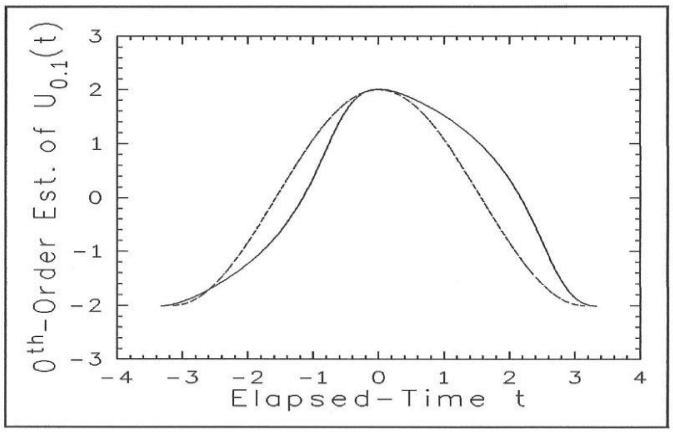


FIGURE 1.3-2 – 1ST-ORDER ESTIMATE

$$U_{1.0}(t) = w_0(\tau) + w_1(\tau)$$

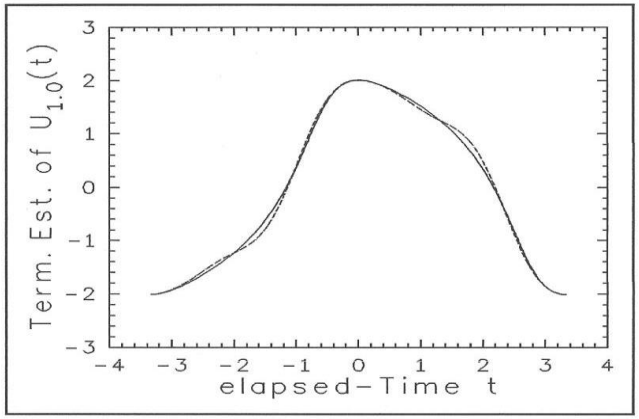


FIGURE 1.3-3 – 2ND-ORDER ESTIMATE

$$U_{1.0}(t) = w_0(\omega\tau) + w_1(\omega\tau) + w_2(\omega\tau)$$

$$[\omega \approx 15/16]$$

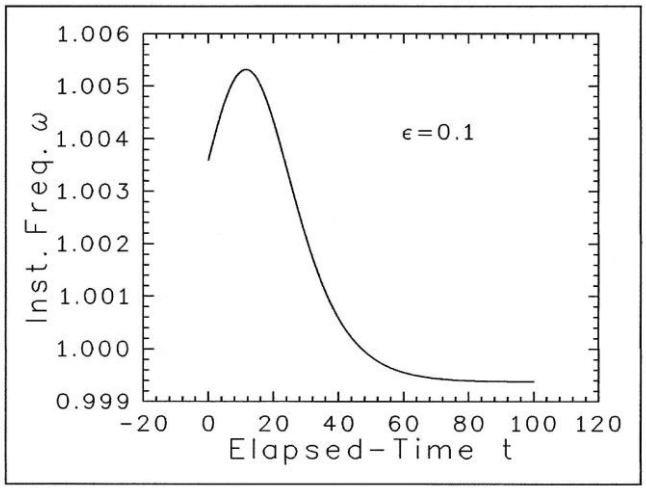


FIGURE 1.3-4 – 2ND-ORDER LVDP SOLUTION
INSTANTANEOUS FREQUENCY ω

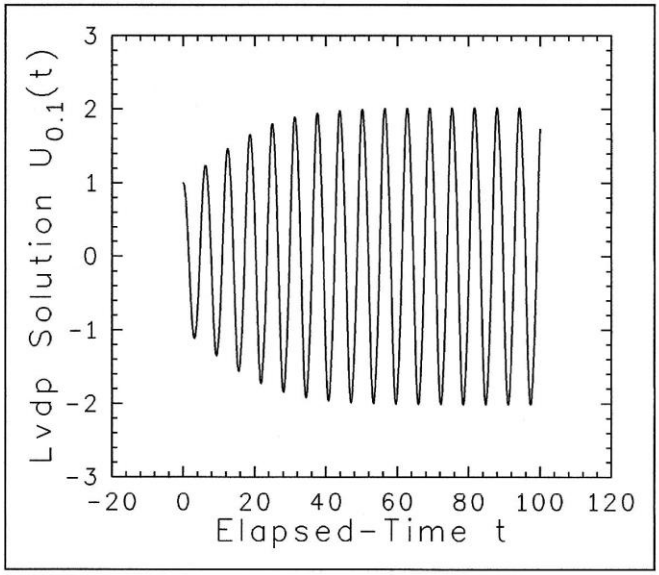
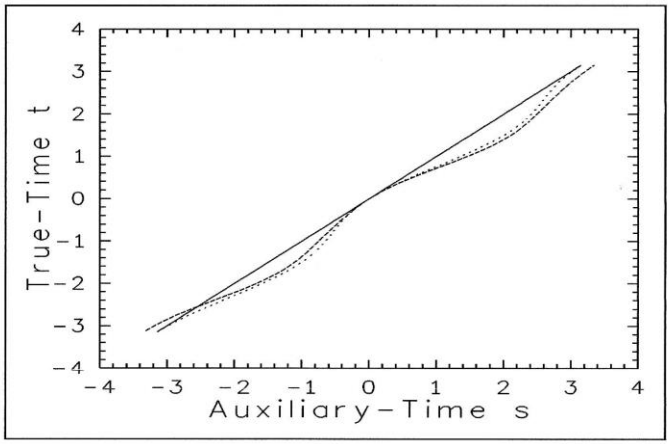


FIGURE 1.3-5 – 2ND-ORDER LVDP SOLUTION $U_{0.1}(t)$



Solid line - 0th Order (no dependence upon ε)
Dotted line - 1st Order (linear dependence upon ε)
Dashed line - 2nd Order (quadratic dependence upon ε)

FIGURE 1.3-6 – EVOLUTION OF $t[\tau]$ AS THE NUMBER OF TERMS IN EQ. (A.2-54) INCREASES

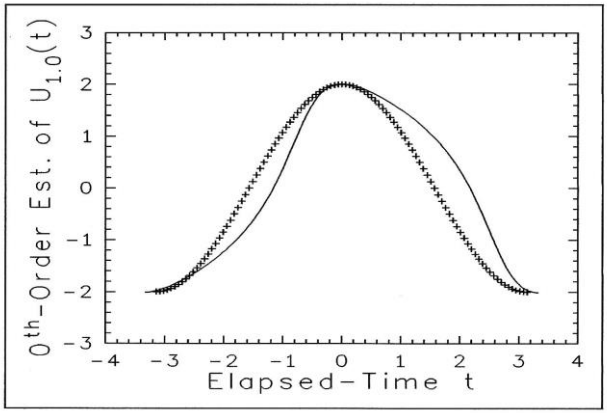


FIGURE 1.3-7 – 0TH ORDER ESTIMATE

$$U_{1.0}(t) = a_{\varepsilon} \cdot Cos(\tau)$$

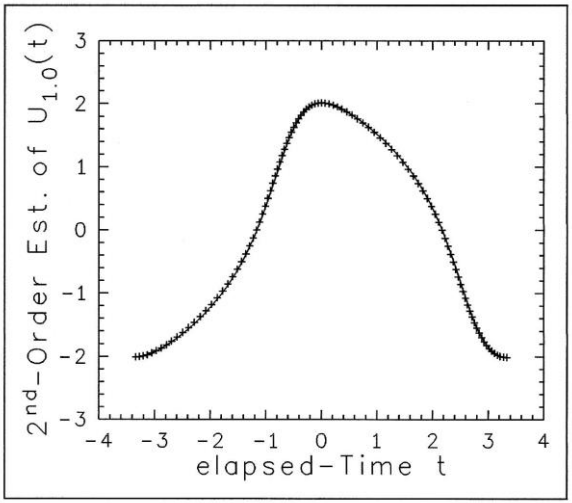


FIGURE 1.3-8 – 1ST ORDER ESTIMATE

$$U_{1.0}(t) = a_{\varepsilon} \cdot Cos(\tau)$$

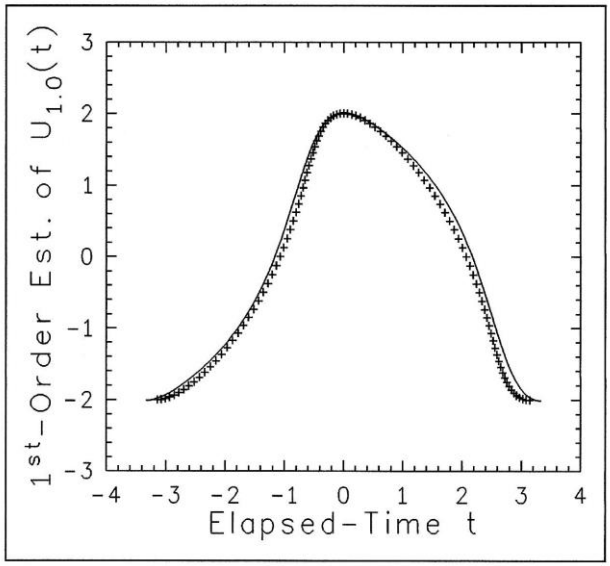


FIGURE 1.3-9 – 2ND ORDER ESTIMATE

$$U_{1.0}(t) = a_{\varepsilon} \cdot Cos(\tau)$$

A.3 ASYMPTOTIC LVDP LIMIT-CYCLE ANALYSES

EQUATION SECTION (NEXT)

The LvdP's asymptotic LvdP Limit-Cycle orbit illustrated in Fig. 1.4-1 satisfies

(A.3-1) $$p[u]\cdot\frac{dp[u]}{du}=\varepsilon\cdot(1-u^2)\cdot p[u]-u$$

whose odd symmetry

(A.3-2) $$p[-u]\equiv -p[u]$$

shows the need to recover only the asymptotic LvdP Limit-Cycle orbit for positive values of momentum, i.e. within Regions R_I and R_{II}. In addition, upon introducing the representation

$$q[u]\equiv\lim_{\varepsilon\to\infty}\frac{p_\varepsilon[u]}{\varepsilon}\ , \tag{A.3-3}$$

it follows from eq. (A.3-1) that $q[u]\to 0$ within Region R_I; in a similar way, it follows that within Region R_{II}, eq. (A.3-1) now assumes the form

$$\frac{dq[u]}{du}=1-u^2 \tag{A.3-4}$$

with its particular solution

$$q[u]=\frac{(u+1)^2\cdot(2-u)}{3} \tag{A.3-5}$$

blending seamlessly with the Region R_I and Region R_{III} solutions $q[u]=0$ at their respective common intersections $u=-1$ and $u=2$. The corresponding Limit Cycle for large but finite values of the nonlinearity parameter ε cannot be recovered directly because the solution of eq. (A.3-4) becomes unbounded in the neighborhood of $u^2=1$ and the Limit Cycle slopes are discontinuous in the neighborhood of $u=\pm 2$.

DORODNICYN [1947] first provided the necessary means for resolving these difficulties by creating an additional Region R_A partially overlapping the Regions R_I and R_{II}, plus an additional Region R_C partially overlapping the Regions R_{II} and R_{III} shown in Fig. 1.4-1; this analysis then used the LvdP Limit-Cycle's odd-symmetry to complete its construction. Subsequent updates [URABE, 1960; PONZO & WAX, 1965; ZONNEVELD, 1966] corrected numerical errors and led to the asymptotic results

$$\begin{aligned}
&\mathrm{U}_0[\varepsilon] = 2 + \frac{\alpha}{3}\cdot\varepsilon^{-4/3} - \frac{16}{27}\cdot\frac{Ln(\varepsilon)}{\varepsilon^2} + \\
&\quad + \frac{1}{9}\cdot[3\cdot b_1 - 1 + 2\cdot Ln(2) - 8\cdot Ln(3)]\cdot\varepsilon^{-2} + O(\varepsilon^{-8/3}) \;; \\
&\text{(A.3-6)}\quad \mathrm{T}_1[\varepsilon] \square\ \mathrm{T}_0[\varepsilon]\cdot\varepsilon^{-1} = [2\cdot Ln(2) - 3] + 3\cdot\alpha\cdot\varepsilon^{-4/3} - \frac{2}{3}\cdot\frac{Ln(\varepsilon)}{\varepsilon^2} + \\
&\quad + [-Ln(3) + 3\cdot Ln(2) - \frac{3}{2} + b_0 - 2\cdot d]\cdot\varepsilon^{-2} + O(\varepsilon^{-7/3}) \\
&\qquad \alpha = 2.338107,\ b_1 = 0.1723,\ d = 0.4889.
\end{aligned}$$

BAVINCK & GRASMAN [1973] provided a seamless blending of the Limit Cycle solutions across Region boundaries by further partitioning Region R_C into two partially overlapping Regions R_D and R_E, while also requiring the local Limit Cycle solutions within adjacent Regions to match smoothly; its analysis is, however, too elaborate for inclusion here, and is instead replaced with an *Abbreviated Summary* (AS), illustrating its procedure and quoting significant results of its analysis.

Within the interior of Region R_I, AS represents the LvdP Limit-Cycle momentum as a *direct* function

$$\text{(A.3-7)}\qquad p_I[u;\varepsilon] = \frac{u}{1-u^2}\cdot\varepsilon^{-1}\ .$$

of the LvdP displacement u, satisfying eq. (A.3-1) through terms of order ε^{-1}; in the same way, within the interior of Region II, AS represents the LvdP momentum as a *direct* function

$$\begin{aligned}
p_{II}[u;\varepsilon] &= [-\frac{u^3}{3} + u + c_0]\cdot\varepsilon + c_1\cdot\varepsilon^{-1/3} + \\
&+ [-\frac{1}{u+1} + \frac{2}{3}\cdot\left\{Ln|2-u| - Ln|u+1|\right\} + c_2]\cdot\varepsilon^{-1}\ .
\end{aligned} \qquad \text{(A.3-8)}$$

of the LvdP displacement u, with the constant values c_0, c_1, and c_2 to be defined subsequently. Of particular importance, eqs. (A.3-7) and (A.3-8) also show the LvdP Limit-Cycle momenta associated with Regions R_I and R_{II} to become discontinuous, hence unable to provide a seamless transition, at their common boundary $u = -1$.

The AS, however, obtains a seamless transition between Regions R_I and R_{II} in the neighborhood of $u = -1$ by introducing an intermediate Region R_A, partially overlapping both Regions R_I and R_{II}, within which it expresses the LvdP Limit-Cycle displacement and momentum as *indirect* functions

$$\begin{aligned} u &= -1 + x \cdot \varepsilon^{-2/3}, \\ p[x,\varepsilon] &= Q(x) \cdot \varepsilon^{-1/3} \ , \end{aligned} \tag{A.3-9}$$

of a *local* variable x, with eq. (A.3-1) now assuming a form

$$Q(x) \cdot [\frac{dQ(x)}{dx} - 2 \cdot x] - 1 = -x \cdot [x \cdot Q(x) + 1] \cdot \varepsilon^{-2/3} \ ; \tag{A.3-10}$$

in particular, the representation $Q(x) = Q_0(x) + Q_1(x) \cdot \varepsilon^{-2/3}$, with

$$\begin{aligned} &Q_0(x) \cdot [\frac{dQ_0(x)}{dx} - 2 \cdot x] - 1 = 0 \ , \\ &Q_0^2(x) \cdot \frac{dQ_1(x)}{dx} + Q_1(x) = -[x^2 \cdot Q_0^2(x) + x \cdot Q_0(x)] \ , \end{aligned} \tag{A.3-11}$$

allows the AS to express the Region R_A momentum in a form

$$p_A[x,\varepsilon] = Q_0(x) \cdot \varepsilon^{-1/3} + Q_1(x) \cdot \varepsilon^{-1} \tag{A.3-12}$$

valid through terms of order ε^{-1}.

Upon representing the displacement in the neighborhood of the Region R_I and R_{II} common boundary as

$$u = -1 + r \cdot \varepsilon^{-1/3} \ , \tag{A.3-13}$$

with $0 < \varepsilon^{-1/3} \cdot r \ll 1$, and substituting this representation into eq. (A.3-7), the AS obtains the asymptotic result

$$p_I[r;\varepsilon] \approx -\frac{1}{2 \cdot r} \cdot \varepsilon^{-2/3} + \frac{1}{4} \cdot \varepsilon^{-1} \ ; \tag{A.3-14}$$

valid through terms of order ε^{-1}; in the same way, upon substituting the representation of eq. (A.3-13) into eq.(A.3-8), the AS obtains the asymptotic result

$$\begin{aligned} p_{II}[r;\varepsilon] &= r^2 \cdot \varepsilon^{+1/3} - \frac{r^3}{3} + c_0 \cdot \varepsilon + c_1 \cdot \varepsilon^{-1/3} + \\ &-\frac{1}{r} \cdot \varepsilon^{-2/3} + \left[\frac{2}{3} Ln(3) + \frac{2}{9} \cdot Ln(\varepsilon) - \frac{2}{3} Ln|r| \right] \cdot \varepsilon^{-1} + c_2 \cdot \varepsilon^{-1} \ . \end{aligned} \tag{A.3-15}$$

with eqs. (A.3-14) and (A.3-15) both valid through terms of order ε^{-1}. Upon further setting $x = \varepsilon^{+1/3} \cdot r$ to make the displacement definition of eq. (A.3-13) match that given by the first line of eq. (A.3-9) , the AS uses eq. (A.3-12) to express the Region R_A momentum as

$$p_A[r,\varepsilon] = Q_0(r \cdot \varepsilon^{1/3}) \cdot \varepsilon^{-1/3} + Q_1(r \cdot \varepsilon^{1/3}) \cdot \varepsilon^{-1} \ . \tag{A.3-16}$$

Within its overlap with Region R_I, where $r < 0$, the AS evaluates eq. (A.3-16) by using the asymptotic values of $Q_0(-x)$ and $Q_1(-x)$; in the same way, within its overlap with Region R_{II}, where $r > 0$, the AS evaluates eq. (A.3-16) by using the asymptotic values of $Q_0(+x)$ and $Q_1(+x)$;

The function $Q_0(x)$ has an explicit representation, in terms of a parameter z,

$$\begin{aligned} x &= \frac{A_i'(z)}{A_i(z)} \ , \\ Q_0(x) &= x^2 - z \ , \end{aligned} \tag{A.3-17}$$

with $A_i(z)$ as the Airy function [ABRAMOWITZ & STEGUN; 1965] shown in Fig. 1.4-2; of particular importance, the parameter values $z \in [\alpha_0, \infty]$ generate the entire range of values $x \in [+\infty, -\infty]$. The asymptotic Airy function representations then yield

$$z \to +\infty : \; Q_0(-x) \propto -\frac{1}{2 \cdot x} \; ,$$
$$z \to -\infty : \; Q_0(+x) \propto x^2 - \alpha_0 - \frac{1}{x} \; , \qquad \text{(A.3-18)}$$

while additional analyses (not included here) yield

$$z \to +\infty : \; Q_1(-x) \propto \frac{1}{4} \; ,$$
$$z \to -\infty : \; Q_1(+x) \propto -\frac{1}{3}x^3 + b_1 - \frac{2}{9} \cdot Ln(\varepsilon) - \frac{2}{3} Ln|x| \; . \qquad \text{(A.3-19)}$$

with $b_1 = 0.1723$.

Thus, within the overlap between Regions R_I and R_A, the asymptotic results of eqs. (A.3-18) and (A.3-19) yield

$$p_A[r,\varepsilon] \square -\frac{1}{2 \cdot r} \cdot \varepsilon^{-2/3} + \frac{1}{4} \cdot \varepsilon^{-1} \; , \qquad \text{(A.3-20)}$$

exactly matching the Region R_I LvdP Limit-Cycle momentum defined by eq. (A.3-14). Similarly, within the overlap between Regions R_A and R_{II} „ the asymptotic results of eqs. (A.3-18) and (A.3-19) yield

$$p_A[r;\varepsilon] = r^2 \cdot \varepsilon^{1/3} - \frac{r^3}{3} - \alpha_0 \cdot \varepsilon^{-1/3} - \frac{1}{r} \cdot \varepsilon^{-2/3} +$$
$$+ [b_1 - \frac{2Ln|\varepsilon|}{9} - \frac{2Ln|r|}{3}] \cdot \varepsilon^{-1} \; ; \qquad \text{(A.3-21)}$$

We now make the Region R_{II} momentum $p_{II}[r;\varepsilon]$ given by eq.(A.3-15) exactly match that of eq.(A.3-21) by mandating the constant values

$$c_0 = \frac{2}{3} \; , \; c_1 = -\alpha_0 \; , \; c_2 = b_1 - \frac{4}{9} \cdot Ln(\varepsilon) - \frac{2}{3} \cdot Ln(3) \; ; \qquad \text{(A.3-22)}$$

we also use these constant values to express the Region R_{II} asymptotic LvdP Limit Cycle momentum defined by eq. (A.3-8) in its definitive form as

$$p_{II}[u;\varepsilon] = \frac{(u+1)^2 \cdot (2-u)}{3} \cdot \varepsilon - \alpha_0 \cdot \varepsilon^{-1/3} +$$

(A.3-23)
$$+ \left\{ \begin{array}{c} -\dfrac{1}{u+1} + \dfrac{2}{3}[Ln|2-u| - Ln|u+1| - \\ - Ln(3)] + b_1 - \dfrac{4}{9} \cdot Ln(\varepsilon) \end{array} \right\} \cdot \varepsilon^{-1} \ .$$

Figure 1.4-3 illustrates how the Region R_A momentum $p_A[u;100]$ given by eq.(A.3-16) for the nonlinearity parameter value $\varepsilon = 100$ blends seamlessly with the corresponding Region R_I and R_{II} momenta $p_I[u;100]$ and $p_{II}[u;100]$ defined by eqs. (A.3-7) and (A.3-9), with circles showing exact values of momentum $p[u;100]$.

We similarly obtain a smooth transition between Regions R_{II} and R_{III} in the neighborhood of their common boundary $p_{II}[u_0,\varepsilon] = 0$, with eq. (A.3-23) yielding

(A.3-24)
$$u_0 = 2 - \frac{\alpha_0}{3} \cdot \varepsilon^{-4/3} \ .$$

We then represent the LvdP Limit-Cycle displacement and momentum through terms of order ε^{-2} as *indirect* functions

(A.3-25)
$$\begin{aligned} p_D[s,\varepsilon] &= (s - s_0) \cdot \varepsilon^{-1}, \\ u(s) &= u_0 + \varepsilon^{-2} \cdot \mathrm{X}_2(s) \ , \end{aligned}$$

of a *local* variable s, valid through terms of order ε^{-2}, with eq. (A.3-1) now assuming the form

(A.3-26)
$$\left\{ (s - s_0) \cdot (u_0^2 - 1) + u_0 \right\} \cdot \frac{d\mathrm{X}_2(s)}{ds} + (s - s_0) = 0 \ .$$

The definition

$$s_0 \equiv \frac{u_0}{u_0^2 - 1} = \frac{2}{3} + \frac{5}{27} \alpha_0 \cdot \varepsilon^{-4/3} \qquad \text{(A.3-27)}$$

then allows us to express eq. (A.3-26) in the simpler form

(A.3-28) $$3 \cdot s \cdot \frac{d\mathrm{X}_2(s)}{ds} + (s - \frac{2}{3}) = 0$$

with the solution

$$\mathrm{X}_2(s) = -\frac{s}{3} + \frac{2}{9} \cdot Ln|s| + d_2(\varepsilon) \quad . \tag{A.3-29}$$

We demonstrate the smooth blending of LvdP momentum within the overlap between Regions R_{II} and R_D by expressing LvdP Limit-Cycle displacement in the form

$$u = 2 - \chi \cdot \varepsilon^{-1} \tag{A.3-30}$$

with eq. (A.3-23) now yielding

(A.3-31) $$\begin{aligned} p_{II}[\chi;\varepsilon] &= 3 \cdot \chi - \alpha_0 \cdot \varepsilon^{-1/3} + \\ &+ \left\{ \begin{array}{l} -\frac{1}{3} + \frac{2}{3} \cdot Ln|\chi| - \\ -\frac{10}{9} \cdot Ln(\varepsilon) - \frac{4}{3} \cdot Ln(3) + b_1 \end{array} \right\} \cdot \varepsilon^{-1} \end{aligned}$$

through terms of order ε^{-1}; similarly, upon inverting the first line of eq. (A.3-25) to give $s = \varepsilon \cdot p_D + s_0$ and substituting the definition of eq. (A.3-29) into the second line of eq.(A.3-25), we obtain

(A.3-32) $$\begin{aligned} \chi &= \frac{1}{3} p_D + \frac{1}{3} \alpha_0 \cdot \varepsilon^{-1/3} + \\ &+ \left\{ \frac{2}{9} - \frac{2}{9} \cdot Ln(p_D) - \frac{2}{9} \cdot Ln(\varepsilon) - d_2(\varepsilon) \right\} \cdot \varepsilon^{-1} \quad . \end{aligned}$$

Thus, upon substituting the definition of χ as given by eq. (A.3-32) into eq. (A.3-31), we obtain

(A.3-33) $$p_{II}[\chi;\varepsilon] = p_D + \left\{ \frac{1}{3} - \frac{16}{9} \cdot Ln(\varepsilon) - 2 \cdot Ln(3) + b_1 - 3 \cdot d_2(\varepsilon) \right\} \cdot \varepsilon^{-1} \ ;$$

the definitions

(A.3-34) $$d_2(\varepsilon) = \frac{1}{9} - \frac{16}{27} \cdot Ln(\varepsilon) - \frac{2}{3} \cdot Ln(3) + \frac{b_1}{3}$$

now guarantee $p_{II}[\chi;\varepsilon] = p_D$. It further follows from eq. (A.3-23) that upon setting $s = s_0$, we obtain $p_D = 0$, thereby recovering

(A.3-35) $$U_0[\varepsilon] = 2 - \frac{\alpha_0}{3} \cdot \varepsilon^{-4/3} + \left[\begin{array}{c} -\frac{1}{9} + \frac{2}{9} \cdot Ln(2) - \frac{8}{9} \cdot Ln(3) - \\ -\frac{16}{27} \cdot Ln(\varepsilon) + \frac{b_1}{3} \end{array} \right] \cdot \varepsilon^{-2}$$

as the Maximum Displacement through terms of order ε^{-2}.

Although eq. (A.3-29) provides an accurate asymptotic expression for the Maximum Displacement through terms of order ε^{-2}, the Region R_D representation given by eq. (A.3-25) does not blend smoothly with the Region R_{III} representation $p_{III}[u;\varepsilon] \equiv -p_I[u;\varepsilon]$ because the displacement representation given by eq. (A.3-29) has a logarithmic singularity in the neighborhood of momentum $p_D = -s_0 \cdot \varepsilon^{-1}$. We resolve this difficulty (without derivations) by introducing an additional Region R_E, overlapping both Region R_D and Region R_{III}, within which we represent the LvdP Limit-Cycle displacement and momentum as indirect functions

(A.3-36) $$\begin{aligned} u &= u_\varepsilon + [x_2(\varepsilon) + \xi] \cdot \varepsilon^{-2}, \\ p_E[\xi;\varepsilon] &= -r_0 \cdot \varepsilon^{-1} + \eta_2(\xi) \cdot \varepsilon^{-2} \end{aligned}$$

of a local variable ξ. The choices

$$\begin{aligned} \eta_2(\xi) &= \frac{5}{9} \cdot \xi + [\frac{10}{81} + \frac{5}{9} \cdot x_2(\varepsilon)] \cdot [1 - e^{+9\xi/2}], \\ x_2(\xi) &= -\frac{28}{27} \cdot Ln(\varepsilon) + \frac{1}{9} - \frac{2}{3} \cdot Ln(3) + \frac{1}{3} \cdot b_1 + \frac{2}{9} Ln[\frac{10}{81} + \frac{5}{9} \cdot x_2(\varepsilon)], \end{aligned} \quad \text{(A.3-37)}$$

then yield a LvdP Limit-Cycle within Region R_E blending smoothly with the LvdP Limit-Cycles within both Regions R_D and R_E.

Figure 1.4-4 illustrates how the Region R_D momentum $p_D[u;10]$ given by eq. (A.3-25) for the nonlinearity parameter value $\varepsilon = 10$ blends

seamlessly with the corresponding Region R_{II} momentum $p_{II}[u;10]$ defined by eq. (A.3-9), with circles showing exact values of momentum $p[u;10]$. Figure 1.4-5 shows, in greater detail, how the Region R_D momentum $p_D[u;10]$ fails to blend smoothly with the Region R_{III} momentum ; it also shows how the Region R_E momentum given by eq. (A.3-36) blends seamlessly with the Region R_D momentum $p_D[u;10]$ and the Region R_{III} momentum $p_{III}[u;10] = -p_I[-u;10]$, with the circles showing exact values of $p[u;10]$; it shows, in particular, the departure of the true LvdP Limit-Cycle momentum from the solid-line representation $p_{II}[u;10]$ provided by eq. (A.3-23) towards the dotted-line representation of $p_D[\chi;\varepsilon]$ as provided by eq. (A.3-31) in the neighborhood of Maximum Displacement $\mathrm{U}_0[10] = 2.01429$ and dot-dash-dot LvdP Limit-Cycle within Region R_E blending smoothly with both the dotted-line LvdP Limit-Cycle of Region R_D and the true Region R_{III}, with the circles again showing exact values of $p[u;10]$.

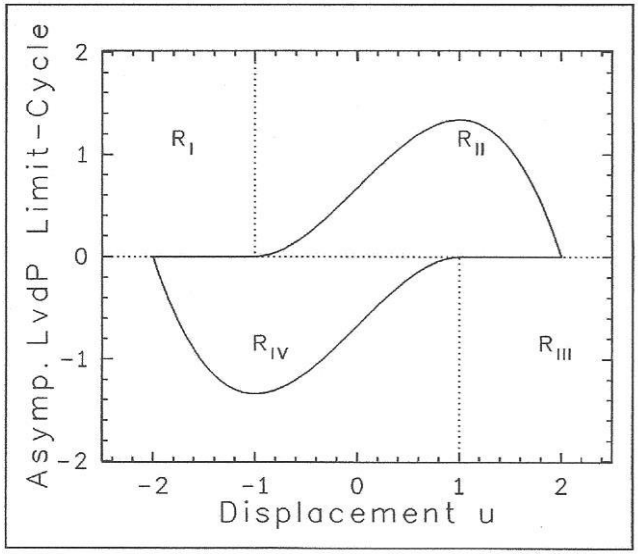


FIGURE 1.4-1 – PERIODIC LVDP SOLUTION'S ASYMPTOTIC LIMIT-CYCLE

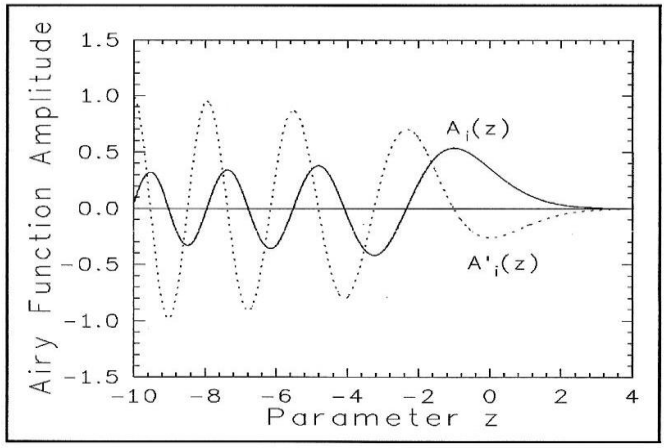


FIGURE 1.4-2 – AIRY FUNCTIONS $A_i(z)$ AND $A_i'(z) = dA_i(z)/dz$

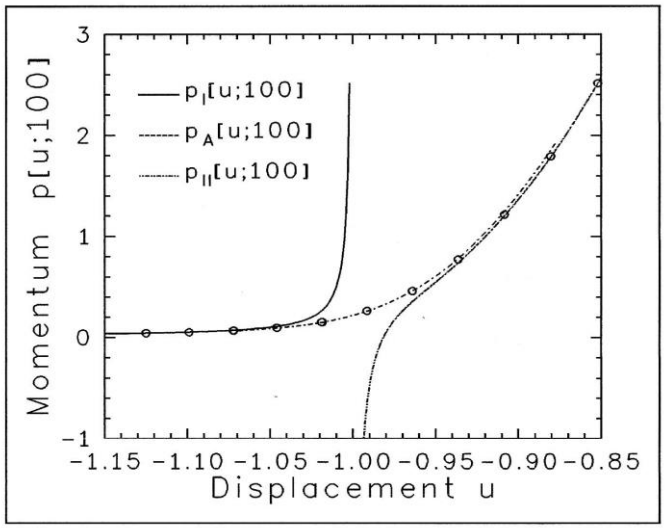


FIGURE 1.4-3 – LVDP LIMIT CYCLE MOMENTA IN THE NEIGHBORHOOD OF DISPLACEMENT $u = -1$

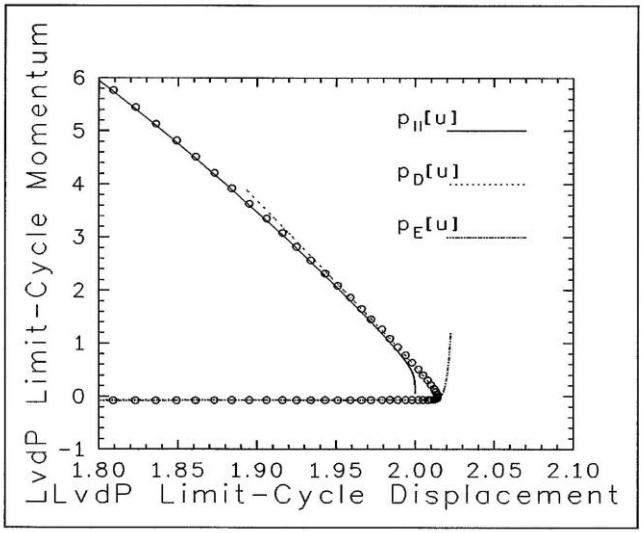


FIGURE 1.4-4 – LVDP LIMIT CYCLE MOMENTA IN THE NEIGHBORHOOD OF MAXIMUM DISPLACEMENT u_0

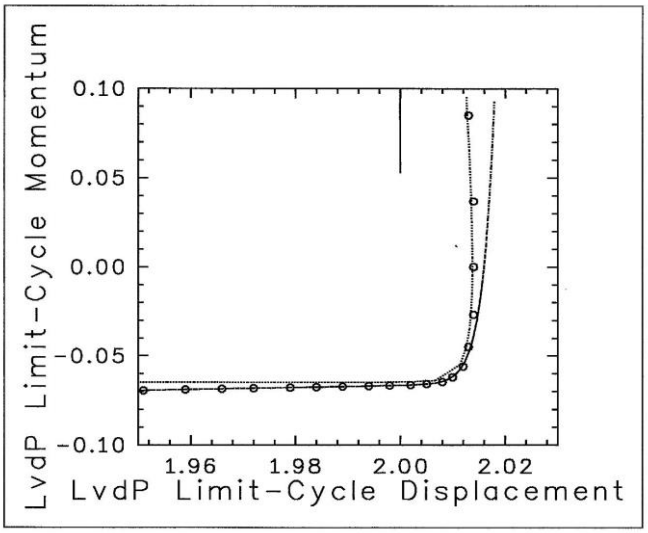


FIGURE 1.4-5 – LVDP LIMIT CYCLE MOMENTA IN THE NEIGHBORHOOD OF MAXIMUM DISPLACEMENT u_0 IN GREATER DETAIL

A.4 OTHER LVDP ANALYSIS METHODS

A.4 ASYMPTOTIC LVDP LIMIT-CYCLE ANALYSES

EQUATION SECTION (NEXT)

THE HOMOTOPY ANALYSIS METHOD

ABBASBANDY ET. AL [2008] use homotopy -- a topological term describing the continuous deformation of one object into another (e.g. a cube into a sphere) -- as an analysis method [HAM] to morph a periodic NF solution $u_N(\tau)$ of the *linear* ODE eq. **Error! Reference source not found.** into a periodic LvdP solution $U_\varepsilon(t)$ of the *nonlinear* ODE eq. **Error! Reference source not found.**. The HAM does this by using the LvdP amplitude A_ε and frequency Ω_ε to express eq. **Error! Reference source not found.** in terms of an of auxiliary-time $\tau \equiv \Omega_\varepsilon \cdot t$ operating upon a *normalized* periodic waveform

$$\phi[\tau + 2\pi] = \phi[\tau] \ , \qquad \phi[0] = 1 \ , \ \frac{d\phi[\tau]}{d\tau} \ ; \tag{A.4-1}$$

in particular, it expresses the LvdP eq. **Error! Reference source not found.** as

$$\Omega_\varepsilon^2 \cdot \frac{d_\varepsilon^2 \phi[\tau]}{d\tau^2} + \varepsilon \cdot \left[A_\varepsilon^2 \cdot \phi_\varepsilon^2[\tau] - 1 \right] \cdot \Omega_\varepsilon \cdot \frac{d\phi_\varepsilon[\tau]}{d\tau} + \phi_\varepsilon[\tau] = 0 \tag{A.4-2}$$

and the NF eq. **Error! Reference source not found.** as

$$\frac{d_\varepsilon^2 \phi_\varepsilon[\tau]}{d\tau^2} + \phi_\varepsilon[\tau_\varepsilon] = 0 \ . \tag{A.4-3}$$

The HAM next attaches an *embedding parameter* $q \in [0,1]$ to the LvdP amplitude A_ε, frequency Ω_ε, and normalized waveform $\phi[\tau]$, while interpreting eq. (A.4-2) as the nonlinear operator

$$N\{\phi_\varepsilon[\tau;q], A_\varepsilon(q), \Omega_\varepsilon(q)\} \equiv$$
$$\equiv \left(\begin{array}{l} \Omega_\varepsilon^2(q) \cdot \dfrac{\partial^2 \phi_\varepsilon[\tau;q]}{\partial \tau^2} + \phi_\varepsilon[\tau;q] \\ +\varepsilon \cdot [A_\varepsilon^2(q) \cdot \phi_\varepsilon^2[\tau;q] - 1] \cdot \Omega_\varepsilon(q) \cdot \dfrac{\partial \phi_\varepsilon[\tau;q]}{\partial \tau} \end{array} \right); \tag{A.4-4}$$

in the same way, upon defining $A_\varepsilon(0) = a_0$ and $\Omega_\varepsilon(0) = 1$, it interprets eq. (A.4-3) as the linear operator

$$L\{\phi_\varepsilon[\tau;q]\} \equiv \left(\frac{\partial^2 \phi_\varepsilon[\tau;q]}{\partial \tau^2} + \phi_\varepsilon[\tau;q] \right); \tag{A.4-5}$$

it then defines the Homotopy

$$\begin{aligned} \mathrm{H}\{\phi_\varepsilon[\tau;q], A_\varepsilon(q), \Omega_\varepsilon(q)\} \equiv (1-q) \cdot L\{\phi_\varepsilon[\tau;q]\} \; - \\ - h \cdot q \cdot N\{\phi_\varepsilon[\tau;q], A_\varepsilon(q), \Omega_\varepsilon(q)\} = 0 \;, \end{aligned} \tag{A.4-6}$$

with h as a *convergence parameter*. Thus, as the embedding parameter increases from an initial value $q = 0$ towards a terminal value $q = 1$, the solution of eq. (A.4-6) morphs from the NF solution $u_{\mathrm{N}}(t) = a_0 \cdot Cos(t)$ of eq. **Error! Reference source not found.** to the periodic LvdP solution $U_0(t) = A_\varepsilon(1) \cdot \phi_\varepsilon[\Omega_\varepsilon(1) \cdot t, 1]$ of eq. **Error! Reference source not found.**.

The HAM recovers the functions $\phi_\varepsilon[\tau;q]$, $A_\varepsilon(q)$, and $\Omega_\varepsilon(q)$ by *presuming* the existence of a value for the *convergence parameter* h permitting their representation as the convergent Maclaurin series

$$\begin{aligned} &\phi_\varepsilon[(\tau;q] = Cos(\tau) + \sum_{m=1}^{\infty} w_m(\tau) \cdot q^m, \\ &A_\varepsilon(q) = a_0 + \sum_{m=1}^{\infty} a_m \cdot q^m \;,\; \Omega_\varepsilon(q) = 1 + \sum_{m=1}^{\infty} \omega_m \cdot q^m, \\ &w_m(\tau) = \frac{1}{m!} \cdot \frac{\partial^m \phi_\varepsilon[\tau;q]}{\partial q^m}\Big|_{q=0}, \\ &a_m = \frac{1}{m!} \cdot \frac{\partial^m A_\varepsilon(q)}{\partial q^m}\Big|_{q=0} \;,\; \omega_m(\tau) = \frac{1}{m!} \cdot \frac{\partial^m \Omega_\varepsilon(q)}{\partial q^m}\Big|_{q=0} \;, \end{aligned} \tag{A.4-7}$$

subject to the initial conditions

(A.4-8) $$w_m(0)=0\ ,\ \frac{\partial w_m(\tau)}{\partial \tau}\Big|_{\tau=0}=0\ ;$$

it then substitutes the representations of eq. (A.4-7) into the Homotopy eq. (A.4-6) to obtain the values of a_m, ω_m, and $w_m(\tau)$ for each power m of the embedding parameter recursively.

The HAM recovers the $m=1$ relationship

(A.4-9) $$\begin{aligned}\mathrm{L}\{w_1(\tau)\}&=\frac{\partial^2 w_1(\tau)}{\partial \tau^2}+w_1(\tau)=h\cdot N\{Cos(\tau),a_0,1\}=\\&=h\cdot\varepsilon\cdot\left[\frac{3\cdot a_0^2}{4}Sin(3\tau)-[\frac{a_0^2}{4}-1]\cdot Sin(\tau)\right],\end{aligned}$$

whence it precludes secular terms in $w_1(\tau)$ by requiring $a_0=2$ and defining

$$w_1(\tau)=h\cdot\varepsilon\cdot[Sin(3\cdot\tau)-3\cdot Sin(\tau)] \qquad \text{(A.4-10)}$$

to satisfy the boundary conditions of eq. (A.4-8). In the same way, the HAM obtains the $m=2$ relationship

$$\begin{aligned}\mathrm{L}\{w_2(\tau)\}&=\frac{\partial^2 w_2(\tau)}{\partial \tau^2}+w_2(\tau)=h\cdot N\{w_1(\tau),a_1,\omega_1\}=\\&=h\cdot\begin{bmatrix}-\varepsilon\cdot a_1\cdot Sin(\tau)+[h\cdot\varepsilon^2\cdot\frac{1}{8}-2\cdot\omega_1]\cdot Cos(\tau)-\\-\varepsilon\cdot[h+a_1+\omega_1]\cdot Sin(3\cdot\tau)-h\cdot\varepsilon^2\cdot\frac{3}{4}\cdot Cos(3\cdot\tau)+\\+h\cdot\varepsilon^2\cdot\frac{5}{8}\cdot Cos(5\cdot\tau)\end{bmatrix},\end{aligned} \qquad \text{(A.4-11)}$$

whence it precludes secular terms in $w_2(\tau)$ by requiring

$$a_1=0\ ,\ \omega_1=h\cdot\varepsilon^2\cdot\frac{1}{16}\ . \qquad \text{(A.4-12)}$$

and defining

$$w_2(\tau) = \hbar^2 \cdot \varepsilon^2 \cdot \frac{1}{192} \cdot [-12 \cdot Cos(\tau) + 18 \cdot Cos(3 \cdot \tau) - 5 \cdot Cos(5 \cdot \tau)] \quad \text{(A.4-13)}$$

to satisfy the boundary conditions of eq. (A.4-8). Upon further requiring

$$\begin{aligned} a_2 &= \hbar^2 \cdot \varepsilon \cdot \frac{1}{96} \ , \\ \omega_2 &= \hbar \cdot \varepsilon \cdot \frac{1}{8} + \hbar^2 \cdot \varepsilon^2 \cdot \frac{1}{16} + \hbar^2 \cdot \varepsilon^4 \cdot \frac{3}{512} \ , \end{aligned} \quad \text{(A.4-14)}$$

to preclude secular terms in the third-order representation of $w_3(\tau)$ (not included here), the HAM eventually obtains

$$U_\varepsilon(t) = A_2 \cdot \begin{bmatrix} Cos(\Omega_2 \cdot t) + \hbar \cdot \frac{\varepsilon}{8} \cdot [Sin(3 \cdot \Omega_2 \cdot t) - 3 \cdot Sin(\Omega_2 \cdot t)] + \\ +\hbar^2 \cdot \varepsilon^2 \cdot \frac{1}{192} \cdot \begin{bmatrix} -12 \cdot Cos(\Omega_2 \cdot t) + \\ +18 \cdot Cos(3 \cdot \Omega_2 \cdot t) \\ -5 \cdot Cos(5 \cdot \Omega_2 \cdot t) \end{bmatrix} \end{bmatrix},$$

(A.4-15)

$$\begin{aligned} \mathrm{A}_2 &= 2 + \frac{\hbar^2 \cdot \varepsilon^2}{96} \ , \\ \Omega_2 &= 1 + \frac{\hbar \cdot \varepsilon^2}{8} + \frac{\hbar^2 \cdot \varepsilon^2}{16} + \frac{3 \cdot \hbar^2 \cdot \varepsilon^4}{512} \ , \end{aligned}$$

with the particular convergence parameter value $\hbar = -1$ (which can not be recovered using the Empirical HAM Recovery rule) making eq. **Error! Reference source not found.** replicate the results of Shohat's analysis, given by eqs. (A.2-24)and (A.2-25) through terms in ε^2.

Continuing in this way, the HAM obtains, after M iterations, the estimates

(A.4-16)

$$\begin{aligned} &\phi_\varepsilon[\tau;1] = \Phi_M(\tau) \square Cos(\tau) + \sum_{m=1}^{M} w_m(\tau) \ , \\ &A_\varepsilon(1) = A_M \square a_0 + \sum_{m=1}^{M} a_m \ , \ \Omega_\varepsilon(1) = \Omega_M \square 1 + \sum_{m=1}^{M} \omega_m \ ; \end{aligned}$$

it then invokes the

EMPIRICAL HAM RECOVERY RULE: Given the M-th order estimates of eq. **Error! Reference source not found.**, the HAM presumes these estimates to be valid when they remains essentially unchanged over a range of convergence parameter values with $h_1 \le h \le h_2$.

Figure 5.1-1 shows the HAM estimates of LvdP frequency Ω_M for the values $M = 2$ and $M = 4$, associated with the particular nonlinearity parameter value $\varepsilon = 1$. Although the (parabolic) data for Ω_2 provides no useful results because there exists no region wherein its value remains essentially unchanged, the (quartic) data for Ω_4 provides a moderate size region $h \in (-1.3, -0.5)$ wherein its value remains essentially unchanged ($\Omega_4 \approx 0.94$ yields a period $T_0[1] = 2\pi/\Omega_4 \approx 6.684$ I in good agreement with the correct value $T_0[1] = 6.663$ given in Table 1.1-1)

THE MULTIPLE TIME SCALE METHOD

The Multiple Time Scale Method [JACOBSEN, 2014] recovers the Averaging Method results in a different way by representing the displacement solution of eq. **Error! Reference source not found.** as a function $u_\varepsilon(s,\tau)$ depending upon a 'standard' elapsed-time, s with $\dot{s} = 1$, and a 'slow' time τ, with $\dot{\tau} = \varepsilon \ll 1$, such that

$$\begin{aligned} \dot{u}_\varepsilon(s,\tau) &= \frac{\partial u_\varepsilon(s,\tau)}{\partial s} + \varepsilon \cdot \frac{\partial u_\varepsilon(s,\tau)}{\partial s}, \\ \ddot{u}_\varepsilon(s,\tau) &= \frac{\partial^2 u_\varepsilon(s,\tau)}{\partial s^2} + \varepsilon \cdot 2 \cdot \frac{\partial^2 u_\varepsilon(s,\tau)}{\partial s \partial \tau} + \varepsilon^2 \cdot \frac{\partial^2 u_\varepsilon(s,\tau)}{\partial \tau^2} \ ; \end{aligned} \tag{A.4-17}$$

it then continues by expressing the LvdP solution in the form

$$u_\varepsilon(s,\tau) = w_0(s,\tau) + \varepsilon \cdot w_1(s,\tau) + \varepsilon^2 \cdot w_2(s,\tau) + \ldots , \tag{A.4-18}$$

substituting the representation of eq. into eq. **Error! Reference source not found.**, and equating terms containing equal powers of the nonlinearity parameter ε, to obtain

$$\frac{\partial^2 w_0(s,\tau)}{\partial s^2} + w_0(s,\tau) = 0 \ ,$$

$$\frac{\partial^2 w_1(s,\tau)}{\partial s^2} + w_1(s,\tau) = -2 \cdot \frac{\partial^2 w_0(s,\tau)}{\partial s \partial \tau} - [w_0^2(s,\tau) - 1)\frac{\partial w_0(s,\tau)}{\partial s} \ , \quad \text{(A.4-19)}$$

$$\square \quad \square \quad \square \ .$$

The MTS next uses the definition $\theta(s,\tau) \square s + \phi(\tau)$ and observes that

$$w_0(s,\tau) \square a(\tau) \cdot Cos[\theta(s,\tau)] \qquad \text{(A.4-20)}$$

satisfies the first line of eq.(A.4-19), thereby allowing its second line to be expressed as

$$\frac{\partial^2 w_1(s,\tau)}{\partial s^2} + w_1(s,\tau) = 2 \cdot \begin{bmatrix} \frac{da(\tau)}{d\tau} \cdot Sin[\theta(s,\tau)] - \\ -a(\tau) \cdot \frac{d\phi(\tau)}{d\tau} \cdot Cos[\theta(s,\tau)] \end{bmatrix} -$$

$$\text{(A.4-21)} \qquad -\left[a^2(\tau) \cdot Cos^2[\theta(s,\tau)] - 1\right] \cdot a(\tau) \cdot Sin[\theta(s,\tau)] =$$

$$= 2 \cdot a(\tau) \cdot \frac{d\phi(\tau)}{d\tau} \cdot Cos[\theta(s,\tau)] +$$

$$+\left[2 \cdot \frac{da(\tau)}{d\tau} + \frac{1}{4} a^3(\tau) - a(\tau)\right] \cdot Sin[\theta(s,\tau)] + \frac{1}{4} a^3(\tau) \cdot Sin[3 \cdot \theta(s,\tau)] \ ;$$

the MTS then precludes secular terms in the solution $w_1(s,\tau)$ by requiring

$$\text{(A.4-22)} \qquad \frac{da(\tau)}{d\tau} = \frac{a(\tau)}{2} \cdot [1 - \frac{a^2(\tau)}{4}] \ , \ \frac{d\phi(\tau)}{d\tau} = 0 \ ,$$

and recovering a solution

$$\text{(A.4-23)} \qquad a(\tau) = \frac{2 \cdot a_0}{\sqrt{a_0^2 + (4 - a_0^2) \cdot e^{-\tau}}} = \frac{2 \cdot a_0}{\sqrt{a_0^2 + (4 - a_0^2) \cdot e^{-\varepsilon \cdot t}}}$$

replicating the results of eq. A.1-5).

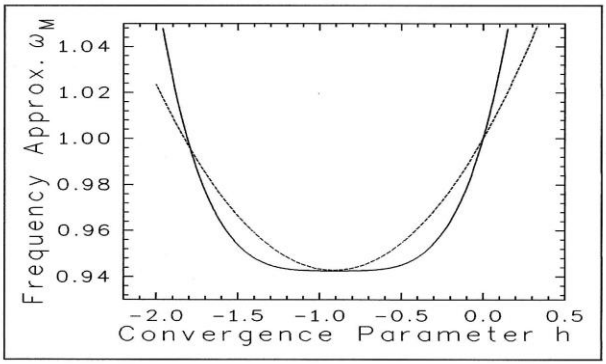


FIGURE 1.5-1 – APPROXIMATE LVDP LIMIT CYCLE FREQUENCY Ω_2 (DOTTED LINE) AND Ω_4 (SOLID LINE) FOR A NONLINEARITY PARAMETER VALUE $\varepsilon = 1$

Annex T – Tables

TABLE 2.4-3 – REPRESENTATIVE VALUES OF $P_1[\gamma;1,m]$ NECESSARY FOR A PERIODIC WD VDPP SOLUTION

	$m=0$	$m=1$	$m=2$	$m=3$	$m=5$	$m=10$
r=0.0	***2.0000***	***2.4495***	***2.8284***	***3.1623***	***3.7417***	***4.8990***
r=0.1	***1.9988***	***2.4476***	***2.8259***	***3.1593***	***3.7378***	***4.8935***
r=0.2	***1.9950***	***2.4418***	***2.8184***	***3.1503***	***3.7262***	***4.8769***
r=0.3	***1.9885***	***2.4318***	***2.8055***	***3.1347***	***3.7062***	***4.8481***
r=0.4	***1.9796***	***2.4174***	***2.7866***	***3.1119***	***3.6765***	***4.8049***
r=0.45	***1.9735***	***2.4084***	***2.7747***	***3.0973***	***3.6574***	**4.7766**
r=0.5	***1.9669***	***2.3979***	***2.7608***	***3.0804***	***3.6349***	**4.7426**
r=0.55	***1.9594***	***2.3860***	***2.7448***	***3.0607***	**3.6084**	**4.7005**
r=0.6	***1.9510***	***2.3724***	***2.7264***	***3.0376***	**3.5761**	**4.6488**
r=0.65	***1.9416***	***2.3569***	***2.7050***	**3.0110**	**3.5357**	**4.5909**
r=0.7	***1.9309***	***2.3399***	***2.6800***	**2.9765**	**3.4884**	**4.5325**
r=0.75	***1.9191***	***2.3189***	**2.6501**	**2.9343**	**3.4405**	**4.4800**
r=0.8	***1.9058***	***2.2952***	**2.6106**	**2.8905**	**3.3979**	**4.4383**
r=0.85	***1.8909***	**2.2748**	**2.5681**	**2.8518**	**3.3649**	**4.4112**

r=0.9	***1.8749***	**2.2314**	**2.5315**	**2.8228**	**3.3452**	**4.4025**
r=0.95	***1.8741***	**2.1887**	**2.5057**	**2.8070**	**3.3415**	**4.4159**
r=1.0	***1.8331***	**2.1561**	**2.4942**	**2.8079**	**3.3586**	**4.4565**
r=1.05	***1.8076***	**2.1385**	**2.5008**	**2.8296**	**3.4007**	**4.5320**
r=1.1	***1.7770***	**2.1403**	**2.5310**	**2.8786**	**3.4772**	**4.6541**
r=1.15	**1.7359**	**2.1681**	**2.5935**	**2.9658**	**3.6010**	**4.4816**
r=1.20	**1.7024**	**2.2341**	**2.7052**	**3.1107**	**3.7969**	**5.1297**
r=1.25	**1.7113**	**2.3633**	**2.8988**	**3.3532**	**4.1155**	**5.5874**
r=1.30	**1.7916**	**2.6187**	**3.2563**	**3.7902**	**4.6799**	**6.3847**
r=1.35	**2.0620**	**3.2463**	**4.0624**	**4.7290**	**5.9133**	**8.1131**
r=1.40	**3.7113**	**6.2762**	**8.0532**	**9.4943**	**11.9044**	**16.4916**
r=1.45	∞	∞	∞	∞	∞	∞

[Normal entries exhibit Libration while bold entries exhibit Rotation. A $P_1[\gamma;d,m]$ value cited as ∞ means the Initial Momentum became infinite for some softness parameter value between that in the cited cell and the previous precious cell]

TABLE 2.4-4 – REPRESENTATIVE VALUES OF $P_1[\gamma;d,0]$ NECESSARY FOR PERIODIC VDPP SOLUTIONS

	$d=1$	$d=2$	$d=3$	$d=5$	$d=10$
Γ=0.0	***2.0000***	***1.6818***	***1.5295***	***1.3763***	**1.2296**
Γ=0.1	***1.9988***	***1.6812***	***1.5291***	***1.3761***	***1.2295***
Γ=0.2	***1.9950***	***1.6796***	***1.5281***	***1.3756***	***1.2293***
Γ=0.3	***1.9885***	***1.6767***	***1.5237***	***1.3747***	***1.2290***
Γ=0.4	***1.9796***	***1.6726***	***1.5239***	***1.3734***	***1.2284***
Γ=0.45	***1.9669***	***1.6671***	***1.5206***	***1.3717***	***1.2277***
Γ=0.5	***1.9510***	***1.6601***	***1.5164***	***1.3695***	***1.2268***
Γ=0.55	***1.9309***	***1.6513***	***1.5112***	***1.3668***	***1.2257***
Γ=0.6	***1.9058***	***1.6404***	***1.5048***	***1.3635***	***1.2244***
Γ=0.65	***1.8741***	***1.6270***	***1.4968***	***1.3595***	***1.2227***
Γ=0.7	***1.8331***	***1.6101***	***1.4870***	***1.3546***	***1.2207***
Γ=0.75	***1.7770***	***1.5887***	***1.4747***	***1.3484***	***1.2183***
Γ=0.8	**1.7359**	***1.5755***	***1.4673***	***1.3448***	***1.2169***
Γ=0.85	**1.7024**	***1.5602***	***1.4589***	***1.3407***	***1.2153***

γ=0.9	**1.7113**	***1.5602***	***1.4491***	***1.3361***	***1.2135***
γ=0.95	**1.7916**	***1.5189***	***1.4377***	***1.3308***	***1.2115***
γ=1.0	**2.0620**	**1.4883**	***1.4237***	***1.3246***	***1.2092***
γ=1.05	**3.7113**	**1.4768**	***1.4062***	***1.3173***	***1.2066***
γ=1.1	∞	**1.5185**	**1.3813**	***1.3083***	***1.2036***
γ=1.5		**1.6965**	**1.3700**	***1.2972***	***1.1999***
γ=1.55		**1.2596**	**1.4144**	**1.2789**	***1.1955***
γ=1.6		∞	**1.6406**	**1.2619**	***1.1899***
γ=1.65			∞	**1.2866**	***1.1823***
γ=1.7				**1.4949**	***1.1709***
γ=1.75				∞	**1.1620**
γ=1.80					**1.2837**
γ=1.85					∞

[Normal entries exhibit Libration while bold entries exhibit Rotation. A $P_1[\gamma;d,m]$ value cited as ∞ means the Initial Momentum became infinite for some softness parameter value between that in the cited cell and the previous precious cell]

ε	δ =0.1	δ =0.2	δ =0.3333	δ =0.5	δ =1.0
0.0	2.99770	2.77628	2.55636	2.35619	2.00000
1.0	2.99829	2.77812	2..56010	2.36208	2.00862
2.0	3.00006	2.78353	2.57070	2.37717	2.01989
3.0	3.00299	2.79225	2.58614	2.39444	2.02330
4.0	3.00705	2.80372	2.60313	2.40785	2.02296
5.0	3.01219	2.81109	2.61850	2.41649	2.02151
6.0	3.01836	2.83133	2.63081	2.42217	2.01983
7.0	3.02545	2.84529	2.64010	2.42249	2.01822
8.0	3.03346	2.85818	2.64698	2.42671	2.01675
9.0	3.04190	2.86957	2.65207	2.42277	2.01544
10.0	3.05093	2.87937	2.65585	2.42828	2.01429
20.0	3.13302	2.92306	2.66642	2.42616	2.00779
30.0	3.17246	2.93250	2.66598	2.42331	2.00516
40.0	3.19182	2.93488	2.66464	2.42138	2.00379
50.0	3.20099	2.93528	2.66341	2.42201	2.00296

60.0	**3.20562**	**2.93505**	**2.66242**	**2.41912**	**2.00240**
70.0	**3.20870**	**2.93462**	**2.66161**	**2.41842**	**2.00201**
80.0	**3.21032**	**2.93413**	**2.66096**	**2.41788**	**2.00172**
90.0	**3.21125**	**2.93366**	**2.66042**	**2.41745**	**2.00150**
100.0	**3.21178**	**2.93322**	**2.66000**	**2.41710**	**2.00132**
∞	**3.20596**	**2.92709**	**2.65564**	**2.41422**	**2.00000**

ε	δ =1.0	δ =2.0	δ =3.0	δ =5.0	δ =10.0
0.0	2.00000	1.68179	1.52945	1.37627	1.22961
1.0	2.00862	1.68418	1.52397	1.36258	1.21232
2.0	2.01989	1.67878	1.51446	1.35353	1.20659
3.0	2.02330	1.67316	1.50864	1.34912	1.20412
4.0	2.02296	1.66904	1.50499	1.34658	1.20278
5.0	2.02151	1.66603	1.50254	1.34496	1.20195
6.0	2.01983	1.66379	1.50079	1.34383	1.20139
7.0	2.01822	1.66206	1.49949	1.34302	1.20099
8.0	2.01675	1.66070	1.49849	1.34241	1.20069
9.0	2.01544	1.65960	1.49770	1.34193	1.20046
10.0	2.01429	1.65870	1.49706	1.34154	1.20028
20.0	2.00779	1.65445	1.49418	1.33987	1.19951
30.0	2.00516	1.65302	1.49326	1.33936	1.19929
40.0	2.00379	1.65233	1.49282	1.33912	1.19919
50.0	2.00296	1.65193	1.49257	1.33899	1.19913
60.0	2.00240	1.65167	1.49242	1.33890	1.19909
70.0	2.00201	1.65149	1.49231	1.33885	1.19907
80.0	2.00172	1.65136	1.49223	1.33881	1.19905
90.0	2.00150	1.65126	1.49217	1.33878	1.19904
100.0	2.00132	1.65118	1.49212	1.33875	1.19903
∞	2.00000	1.65067	1.49182	1.33856	1.19894

ε	δ =0.1	δ =0.2	δ =0.3333	δ =0.5	δ =1.0
0.0	2.99770	2.77628	2.55636	2.35619	2.00000
1.0	3.00226	2.79191	2.59273	2.42393	2.17271
2.0	3.01597	2.83904	2.70236	2.62505	2.61497
3.0	3.03890	2.91818	2.88405	2.93915	3.16872
4.0	3.07117	3.02952	3.12909	3.32424	3.76234
5.0	3.11292	3.17211	3.42093	3.74622	4.37523
6.0	3.16426	3.34314	3.74280	4.18823	4.99943
7.0	3.22525	3.53812	4.08323	4.64245	5.63109
8.0	3.29587	3.75184	4.43566	5.10479	6.26802
9.0	3.37595	3.97948	4.79630	5.57290	6.90889
10.0	3.46517	4.21730	5.16289	6.04529	7.55281
20.0	4.70813	6.82654	8.97289	10.8803	14.0775
30.0	6.21014	9.57172	12.8777	15.7939	20.6677
40.0	7.78187	12.3573	16.8150	20.7363	27.2833
50.0	9.38314	15.1627	20.7687	25.6937	33.9122
60.0	11.0039	17.9795	24.7323	30.6602	40.5493
70.0	12.6276	20.8038	28.7025	35.6328	47.1919
80.0	14.2615	23.6334	32.6773	40.6097	53.8383
90.0	15.9002	26.4268	36.6557	45.5899	60.4877
100.0	17.5426	29.3032	40.6366	50.5726	67.1393
∞	∞	∞	∞	∞	∞

ε	δ =1.0	δ =2.0	δ =3.0	δ =5.0	δ =10.0
0.0	2.00000	1.68179	1.52945	1.37627	1.22962
1.0	2.17271	2002886	1.98412	1.94557	1.90822
2.0	2.61497	2.68421	2.71890	2.74575	2.75740
3.0	3.16872	3.40170	3.50068	3.58330	3.63885
4.0	3.76234	4.14331	4.30283	4.43908	4.53779
5.0	4.37523	4.89811	5.11681	5.30601	5.44784
6.0	4.99943	5.66134	5.93859	6.18048	6.36554
7.0	5.63109	6.43045	6.76593	7.06039	7.28880
8.0	6.26802	7.20389	7.59741	7.94440	8.21624
9.0	6.90889	7.98066	8.43210	8.83158	9.14694
10.0	7.55281	8.76007	9.26934	9.72130	10.0802
20.0	14.0775	16.6307	10.77164	18.6925	19.4882
30.0	20.6677	24.5613	26.2219	27.7211	28.9535
40.0	27.2833	32.5155	34.7501	36.7707	38.4401
50.0	33.9122	40.4819	43.2901	45.8335	47.3744
60.0	40.5493	48.4558	51.8372	54.9021	57.4411
70.0	47.1919	56.4346	60.3890	63.9751	66.9489
80.0	53.8383	64.4169	68.9441	73.0511	76.4596
90.0	60.4877	12.4019	77.5016	82.1295	85.9723
100.0	67.1393	80.3888	86.0610	91.2095	95.4865
∞	∞	∞	∞	∞	∞

ε	δ =0.1	δ =0.2	δ =0.3333	δ =0.5	δ =1.0
0.0	6.28319	6.28319	6.28319	6.28319	6.28319
1.0	6.29204	6.31455	6.35849	6.42709	6.66329
2.0	6.31862	6.40874	6.58329	6.84753	7.62987
3.0	6.36298	6.56560	6.94950	7.49104	8.85909
4.0	6.42519	6.78373	7.43412	8.27167	10.2035
5.0	6.50529	7.05918	8.00293	9.12710	11.6122
6.0	6.60323	7.38482	8.62547	10.0271	13.0619
7.0	6.71882	7.75117	9.28240	10.9566	14.5397
8.0	6.85166	8.14847	9.96280	11.9071	16.0382
9.0	7.00107	8.56851	10.6603	12.8733	17.5522
10.0	7.16612	9.00323	11.3707	13.8517	19.0784
20.0	9.37521	13.7833	18.8246	23.9737	34.6823
30.0	11.9820	18.8403	26.5354	34.3491	50.5437
40.0	14.7078	23.9966	34.3429	44.8202	66.5014
50.0	17.4906	29.2039	42.1991	55.3408	82.5083
60.0	20.3068	34.4421	50.0861	65.8914	98.5448
70.0	23.1451	39.7009	57.9933	76.6191	114.601
80.0	25.9989	44.9744	65.9149	87.0465	130.670
90.0	28.8542	50.2589	73.8471	97.6415	146.750
100.0	31.7385	55.5518	81.7875	108.245	162.837
∞	0.29381	0.53525	0.79974	1.06569	1.61366

ε	δ =1.0	δ =2.0	δ =3.0	δ =5.0	δ =10.0
0.0	6.28319	6.28319	6.28319	6.28319	6.28319
1.0	6.66329	7.06883	7.33650	7.64890	7.98524
2.0	7.62987	8.64627	9.23302	9.89127	10.5856
3.0	8.85909	10.4648	11.3716	12.3830	13.4477
4.0	10.2035	12.3967	13.6266	14.9967	16.4395
5.0	11.6122	14.3942	15.9497	17.6818	19.5076
6.0	13.0619	16.3412	18.3168	20.4133	22.6252
7.0	14.5397	18.5036	20.7145	23.1769	25.7771
8.0	16.0382	20.5945	23.1345	25.9640	28.9541
9.0	17.5522	22.7018	25.5715	28.7688	32.1499
10.0	19.0784	24.8219	28.0216	31.5875	35.3603
20.0	34.6823	46.3795	52.8906	60.1648	67.8651
30.0	50.5437	68.2024	78.0299	88.9995	100.657
40.0	66.5014	90.1226	103.267	117.943	133.549
50.0	82.5083	112.092	128.553	146.935	166.489
60.0	98.5448	134.090	153.868	175.955	199.458
70.0	114.601	156.107	179.201	204.993	232.444
80.0	130.670	178.137	204.548	234.045	265.442
90.0	146.750	200.177	229.904	263.105	298.449
100.0	162.837	222.225	255.266	292.172	331.462
∞	1.61366	2.20928	2.54065	2.91095	3.30525